Coq Summer School, Session 2 : Basic programming with numbers and lists

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Predefined data structures

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```
Inductive bool := true | false.

Inductive nat := O : nat | S : nat -> nat.

Inductive list A :=
 | nil : list A
 | cons : A -> list A -> list A.
```

- Nota: `a::b` is a notation for `(cons a b)`.

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Pattern matching

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- Most common situation: one pattern for each constructor.
- NB: for `bool`, an alternative syntactic sugar is `if b then false else true`.

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Definition pred x :=
 match x with
  | S x' => x'
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 end.

Definition iszero x :=
 match x with
  | O => true
  | S _ => false
 end.
```

More complex pattern matching

- We can use deeper patterns, combined matchings, as well as wildcards:

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Definition istwo x :=
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  | S (S O) => true
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- These matchings are not atomic, but rather expansed internally into nested matchings (use `Print` to see how many).

Recursion

- When using `Fixpoint` instead of `Definition`, recursive sub-calls are allowed (at least some of them).

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Fixpoint div2 n :=
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```
Fixpoint div2 n :=
 match n with
  | S (S n') => S (div2 n')
  | _ => 0
 end.
```

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- This way, termination of computations is (syntactically) ensured.
- Example of rejected recursive functions:

```
Fixpoint loop n := loop (S n).
Fixpoint div2_ko n :=
 if leb n 1 then 0 else S (div2_ko (n-2)).
```

Some other recursive functions over nat

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 end.

Fixpoint beq_nat n m := match n, m with
   | S n', S m' => beq_nat n' m'
   | O, O => true
   | _, _ => false
 end.
```

Recursion over lists

- With recursive functions over lists, the main novelty is *polymorphism* :

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Fixpoint length A (l : list A) :=
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- NB: `(app l1 l2)` is noted `l1++l2`.
- NB: we use here *Implicit Arguments* to avoid writing type parameters such as `A` again and again when applying functions.

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- With `fold_right`, computation starts at the end of the list:
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Eval vm_compute in
 fold_right (fun x l => x::l) nil (1::2::3::nil).
  ==> 1::2::3::nil
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 : A :=
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Eval vm_compute in
 fold_left (fun l x => x::l) nil (1::2::3::nil).
  ==> 3::2::1::nil
```

A complete example: mergesort

- Part I: splitting a list in two
- Part II: merging two sorted lists into one
- Part III: iterating the process...

Mergesort I : splitting

```
Fixpoint split A (l : list A) : list A * list A :=
 match l with
  | nil => (nil, nil)
  | a::nil => (a::nil, nil)
  | a::b::l' => let (l1, l2) := split l' in (a::l1, b::l2)
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 end.

Eval vm_compute in split (1::2::3::4::5::nil).
  ==> (1::3::5::nil, 2::4::nil)
```

Mergesort II : merging

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Definition merge A (less:A->A->bool)
 : list A -> list A -> list A :=
 fix merge l1 := match l1 with
   | nil => (fun l2 => l2)
   | x1::l1' =>
      (fun l2 => match l2 with
        | nil => l1
        | x2::l2' =>
           if less x1 x2 then x1 :: merge l1' l2
                         else x2 :: merge l1 l2'
      end)
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- No structurally decreasing argument: `Rejected!`

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      end)
 end.
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- Trick of the specialized internal fix: `Accepted!`

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Definition mergeloop A (less:A->A->bool) :=
 fix loop (l:list A) (n:nat) :=
  match n with
   | O => nil
   | S n => match l with
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              | _ => let (l1,l2) := split l in
                 merge less (loop l1 n) (loop l2 n)
            end
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```
Definition mergesort A less (l:list A) :=
  mergeloop less l (length l).
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