Finite Developments in the λ-calculus

Part 3

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Labeled lambda-calculus

Exercise 1 Show that residuals of redexes keep same names by case inspection on occurrences of redexes.

Exercise 2 Show that $M \longrightarrow N$ implies $M^\alpha \longrightarrow N^\alpha$

Exercise 3 Show the parallel moves lemma (with Martin-Löf way)

If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.

Exercise 4 Label Y_f, draw its reduction graph and show redexes families when $Y_f = (\lambda x.f(xx))(\lambda x.f(xx))$

Exercise 5 Same with KaY_f

Inside-out reductions

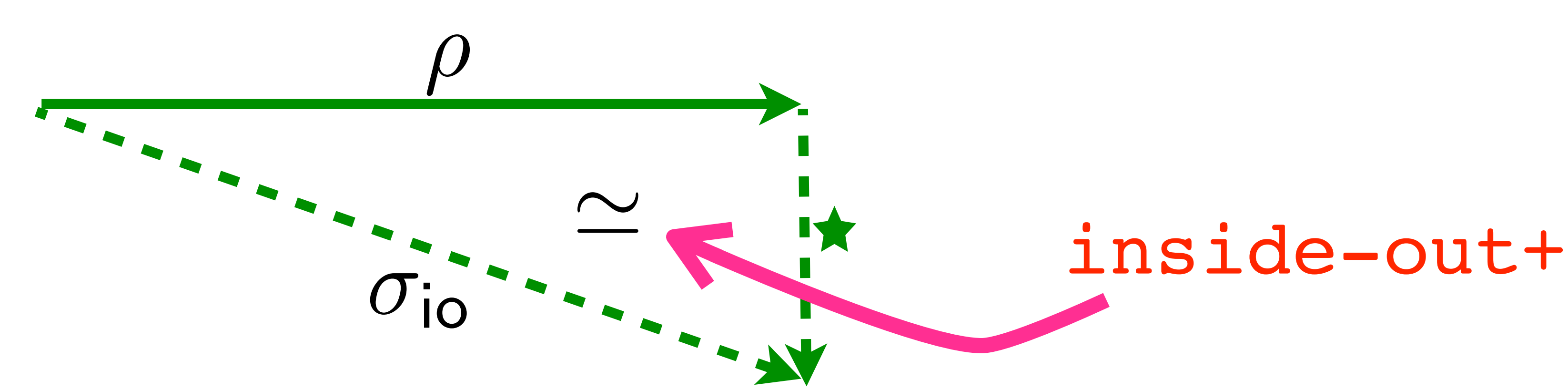
- **Definition:** The following reduction is **inside-out**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j inside R_i in M_{i-1}.

- **Theorem** [Inside-out completeness, 74]

Let $M \xrightarrow{\star} N$. Then $M \xrightarrow[\text{io}]{\star} P$ and $N \xrightarrow{\star} P$ for some P.

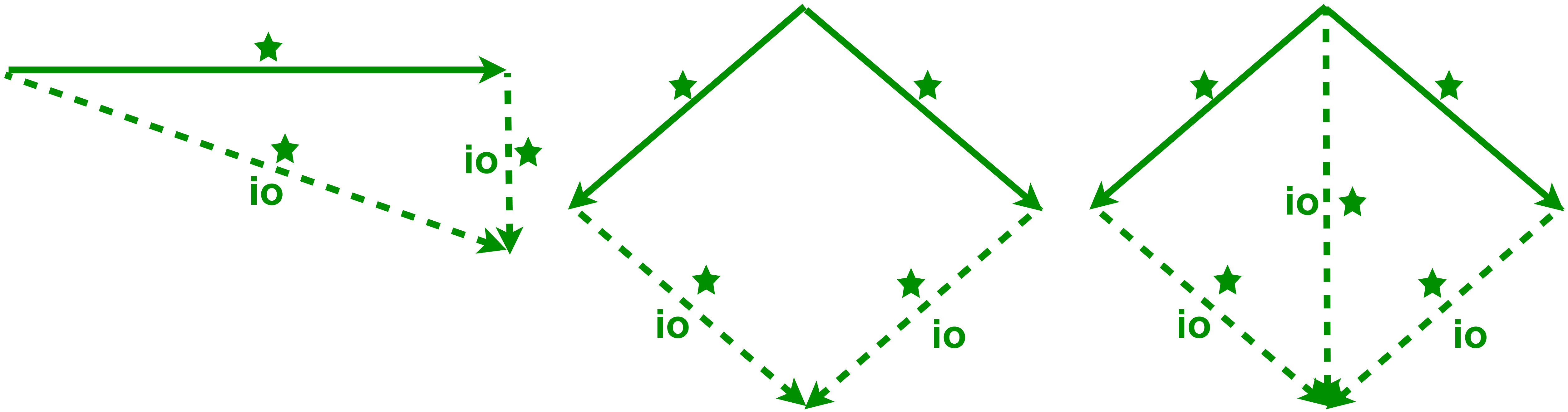


Exercises

Exercise 6 Prove inside-out completeness

Hint: use Finite Development theorem.

Exercise 7 Prove the following diagrams



Permutation equivalence

Proof [uniqueness of labeled standard]

Let ρ and σ be 2 distinct coinitial pure labeled standard reductions.

Take first step when they diverge. Call M that term.
We make structural induction on M. Say ρ is more to the left.
If first step of ρ contracts an internal redex, we use induction.
If first step of ρ contracts an external redex, then:

$$
\begin{tikzcd}
& M = ((\lambda x.P)^{\alpha}\, Q)^{\beta} \arrow[dl] \arrow[ddr, bend left, "\mathbf{st}"] & \\
P^{\lceil\alpha\rceil\beta}\{x := Q^{\lfloor\alpha\rfloor}\} \arrow[d, "\mathbf{st}"] & & \\
N^{\lceil\alpha\rceil\beta} & \neq & ((\lambda x.A)^{\alpha} B)^{\beta}
\end{tikzcd}
$$

Permutation equivalence

- **Corollary** `[labeled prefix ordering]`

  Let $\rho : M \xrightarrow{\star} N$ and $\sigma : M \xrightarrow{\star} P$ be coinitial pure labeled reductions.
  Then $\rho \sqsubseteq \sigma$ iff $N \xrightarrow{\star} P$.

- **Exercise 8** Show the following properties

  (i) $\rho \sqsubseteq \rho$

  (ii) $\rho \sqsubseteq \sigma \sqsubseteq \rho$ implies $\rho \simeq \sigma$

  (iii) $\rho \sqsubseteq \sigma \sqsubseteq \tau$ implies $\rho \simeq \tau$

  (iv) $\rho \sqsubseteq \sigma$ implies $\rho/\tau \sqsubseteq \sigma/\tau$

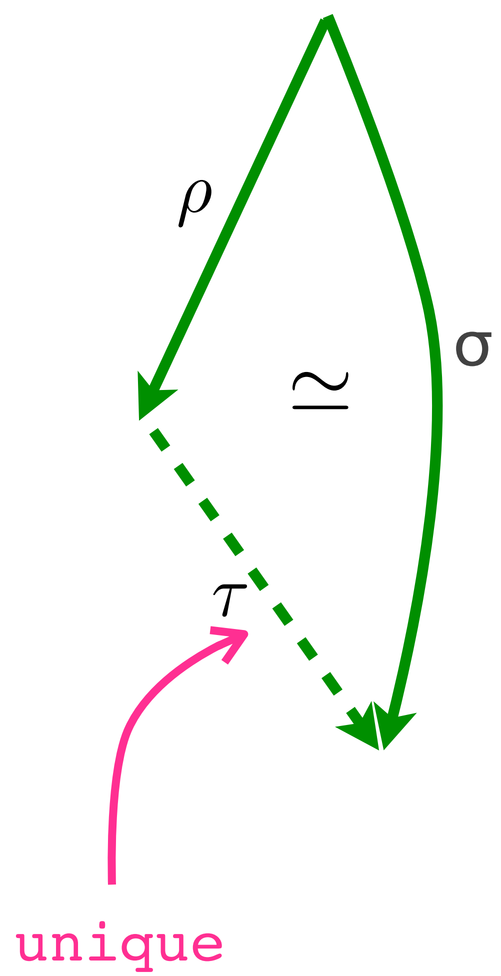
  (v) $\rho \sqsubseteq \sigma$ iff $\exists \tau,\ \rho\tau \simeq \sigma$

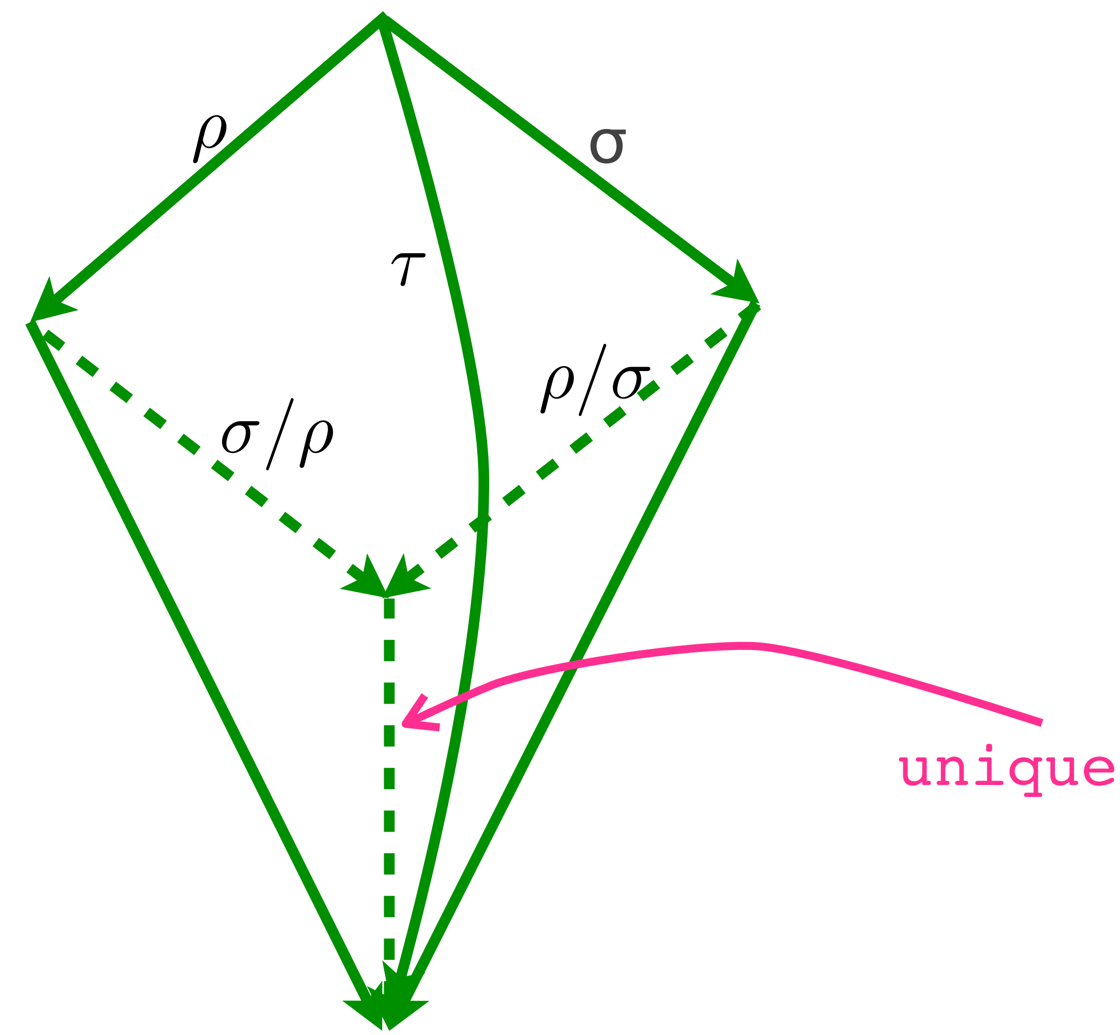
  (vi) $\rho \sqsubseteq \rho \sqcup \sigma$, $\sigma \sqsubseteq \rho \sqcup \sigma$

  (vii) $\rho \sqsubseteq \tau$, $\sigma \sqsubseteq \tau$ implies $\rho \sqcup \sigma \sqsubseteq \tau$

Permutation equivalence

- **Exercise 9** Show the following diagrams





Permutation equivalence

- **Corollary** `[lattice of labeled reductions]`

  Labeled reduction graphs are upwards semi lattices for any pure labeling.

- **Corollary** `[push-out category]`

  Prefix ordering on reductions is a push-out.

- **Exercise 10** Try on $(\lambda x.x)((\lambda y.(\lambda x.x)a)b)$ or $(\lambda x.xx)(\lambda x.xx)$

- **Exercise 11** Show that prefix ordering on reductions is not a pull-back.

Proof of the GFD theorem

Bound on heights of labels

- **Definition** The height of a label is its nesting of underlines and overlines

$$
\begin{aligned}
&h(a) = 0 \\
&h(\lceil\alpha\rceil) = h(\lfloor\alpha\rfloor) = 1 + h(\alpha) \\
&h(\alpha\beta) = \max\{\alpha, \beta\}
\end{aligned}
$$

- **Fact** Let $\mathcal{F}$ be a finite set of redex families, then there is an upper bound $H(\mathcal{F})$ on labels of subterms in reductions relative to $\mathcal{F}$.

When initial term is labeled with atomic letters, we have

$$H(\mathcal{F}) = \max\{h(\alpha) \mid \alpha \in \mathcal{F}\}$$

Proof of finite developments

- **Notation** $\tau(M^\alpha) = \alpha$ when M has an empty external label

- **Lemma 1** Let $M \xrightarrow{\star} M'$, then $h(\tau(M)) \leq h(\tau(M'))$

- **Lemma 2** Let $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n} \xrightarrow{\star} (\lambda x.N)^\alpha$
  Then $h(\tau(M)) \leq h(\alpha)$

- **Lemma 3** [Barendregt] Let $M\{x := N\} \xrightarrow{\star} (\lambda y.P)^\alpha$
  There are 2 cases:

  $M \xrightarrow{\star} (\lambda y.M')^\alpha$ and $M'\{x := N\} \xrightarrow{\star} P$

  $M \xrightarrow{\star} M' = (\cdots((x^\beta\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n}$ and $M'\{x := N\} \xrightarrow{\star} (\lambda y.P)^\alpha$

Proof of finite developments

- **Lemma 1** Let $M \xrightarrow{*} N$, then $h(\tau(M)) \leq h(\tau(N))$

Proof by induction on length of reduction. Let $M \xrightarrow{R} N,\ R = ((\lambda x.A)^{\alpha} B)^{\beta}$

If R is internal in M, then $\tau(M) = \tau(N)$.

If $M = R = ((\lambda x.A)^{\alpha} B)^{\beta} \longrightarrow A\{x := B^{\lfloor \alpha \rfloor}\}^{\lceil \alpha \rceil \beta} = N$,

then $h(\tau(M)) = h(\beta) \leq h(\gamma\beta) = h(\tau(N))$ for some γ.

- **Lemma 2** Let $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_n)^{\beta_n} \xrightarrow{*} (\lambda x.N)^{\alpha}$

Then $h(\tau(M)) \leq h(\alpha)$

Proof by induction on n.

When $n = 0$, obvious by lemma 1.

Otherwise $(\cdots((M\, M_1)^{\beta_1}\, M_2)^{\beta_2} \cdots M_{n-1})^{\beta_{n-1}} \xrightarrow{*} (\lambda y.P)^{\gamma}$

and $((\lambda y.P)^{\gamma} Q)^{\beta_n} \longrightarrow P\{y := Q^{\lfloor \gamma \rfloor}\}^{\lceil \gamma \rceil \beta_n} \xrightarrow{*} (\lambda x.N)^{\alpha}$

So $h(\tau(M)) \leq h(\gamma) < h(\delta\lceil \gamma \rceil \beta_n) \leq h(\alpha)$ by induction and lemma 1.

Proof of finite developments

- **Lemma 3** [Barendregt] Let $M\{x := N\} \xrightarrow{\star} (\lambda y.P)^\alpha$

  There are 2 cases:

  $$M \xrightarrow{\star} (\lambda y.M')^\alpha \text{ and } M'\{x := N\} \xrightarrow{\star} P$$

  $$M \xrightarrow{\star} M' = (\cdots((x^\beta\ M_1)^{\beta_1}\ M_2)^{\beta_2}\cdots M_n)^{\beta_n} \text{ and } M'\{x := N\} \xrightarrow{\star} (\lambda y.P)^\alpha$$

Proof Let $M^* = M\{x := N\}$. There are 3 cases on weak head reduction of M : it reaches an abstraction or a head variable which has to be x.
More precisely, we consider the standard reduction from M^* to $(\lambda y.P)^\alpha$.

Case 1: $M = (\lambda y.M')^\alpha$ and we are done since $M^* = (\lambda y.M'^*)^\alpha$.

Case 2: $M = ((\cdots((y^\beta\ M_1)^{\beta_1}\ M_2)^{\beta_2})\cdots M_n)^{\beta_n}$. Then $y = x$ and $M' = M$.

Case 3: $M = (\cdots((((\lambda z.A)^\beta\ B)^\gamma\ C_1)^{\beta_1}\ C_2)^{\beta_2}\cdots C_n)^{\beta_n}$

Let $M_1 = (\cdots((A\{z := B^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\gamma}\ C_1)^{\beta_1}\ C_2)^{\beta_2}\cdots C_n)^{\beta_n}$

Then $M^* = (\cdots((((\lambda z.A^*)^\beta\ B^*)^{\beta_1}\ C_1^*)^{\beta_1}\ C_2^*)^{\beta_2}\cdots C_n^*)^{\beta_n} \longrightarrow M_1^*$ is the first step of the standard reduction from M^* to $(\lambda y.P)^\alpha$. By induction on its length, we are done.

Proof of finite developments

- **Notation** Let $\mathcal{SN}_{\mathcal{F}}$ be the set of strongly normalizable terms w.r.t. reductions relative to $\mathcal{F}$.

- **Lemma** `[subst]` Let $\mathcal{F}$ be a finite set of redex families.
  $M, N \in \mathcal{SN}_{\mathcal{F}}$ implies $M\{x := N\} \in \mathcal{SN}_{\mathcal{F}}$

  Proof `[van Daalen]` by induction on $\langle H(\mathcal{F}) - h(\tau(N)), \text{depth}(M), \|M\| \rangle$

- **Theorem** `GFD` Let $\mathcal{F}$ be a finite set of redex families.
  Then $M \in \mathcal{SN}_{\mathcal{F}}$ for all M.

  Proof by easy induction on $\|M\|$

Proof of finite developments

- **Lemma** `[subst]` Let $\mathcal{F}$ be a finite set of redex families.
  $M, N \in \mathcal{SN}_{\mathcal{F}}$ implies $M\{x := N\} \in \mathcal{SN}_{\mathcal{F}}$

Proof `[van Daalen]` by induction on $\langle H(\mathcal{F}) - h(\tau(N)), \mathrm{depth}(M), \|M\| \rangle$

Cases $M = x$, $M = y$, $M = \lambda y.M_1$ are obvious or easy by induction on $\|M\|$.

Write M^* for $M\{x := N\}$ and consider case $M = (M_1 M_2)^\alpha$.

If all reductions are internal to M_1^* and M_2^*, then easy induction on $\|M\|$.

Otherwise, let $M_1^* \xrightarrow{\star} (\lambda y.P)^\beta$ and $M_2^* \xrightarrow{\star} Q$ and $((\lambda y.P)^\beta Q)^\alpha \longrightarrow P\{y := Q^{\{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$

Then M_1^* and M_2^* are in $\mathcal{SN}_{\mathcal{F}}$ by induction on $\|M\|$,

and $M_1^* \xrightarrow{\star} (\lambda y.P)^\beta$ and $M_2^* \xrightarrow{\star} Q$. So P and Q are in $\mathcal{SN}_{\mathcal{F}}$.

How is $P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$??

By lemma 3, we have 2 cases:

Proof of finite developments

Case 1:

Then $M_1 \xrightarrow{\star} (\lambda y.M_1')^\beta$ and $M_1'^* \xrightarrow{\star} P$.

Therefore $M_1'^*\{y := M_2^{*\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \xrightarrow{\star} P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$.

But as $M = (M_1 M_2)^\alpha \xrightarrow{\star} ((\lambda y.M_1')^\beta M_2)^\alpha \longrightarrow M' = M_1'\{y := M_2^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$,
we have $\mathrm{depth}(M') < \mathrm{depth}(M)$.

Thus by induction $M'^* = M_1'^*\{y := M_2^{*\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \in \mathcal{SN}_{\mathcal{F}}$
and $P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \in \mathcal{SN}_{\mathcal{F}}$.

$$\begin{tikzcd}
 & (M_1 M_2)^\alpha \arrow[dl, "\star"'] \\
((\lambda y.M_1')^\beta M_2)^\alpha \arrow[d] & \\
M_1'\{y := M_2^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} &
\end{tikzcd}$$

lower depth

$$\begin{tikzcd}
 & (M_1^* \, M_2^*)^\alpha \arrow[dl, "\star"'] \arrow[ddr, "\star"] & \\
((\lambda y.M_1'^*)^\beta M_2^*)^\alpha \arrow[d] \arrow[drr, "\star"] & & \\
M_1'^*\{y := M_2^*\lfloor\beta\rfloor\}^{\lceil\beta\rceil\alpha}, \arrow[drr, "\star"'] & & ((\lambda y.P)^\beta Q)^\alpha \arrow[d] \\
 & & P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}
\end{tikzcd}$$

Proof of finite developments

Case 2:

$M_1 \xrightarrow{\star} M_1' = (\cdots((x^\gamma\, A_1)^{\gamma_1}\, A_2)^{\gamma_2})\cdots A_n)^{\gamma_n}$ and

$$M_1'^{*} = (\cdots((N^\gamma\, A_1^*)^{\gamma_1}\, A_2^*)^{\gamma_2})\cdots A_n^*)^{\gamma_n} \xrightarrow{\star} (\lambda y.P)^\beta$$

Therefore $h(\tau(N)) \leq h(\tau(N^\gamma)) \leq h(\beta)$ by lemma 2.

So $M^* = (M_1^* M_2^*)^\alpha \xrightarrow{\star} ((\lambda y.P)^\beta Q)^\alpha \xrightarrow{\star} P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha}$

and $h(\tau(N)) \leq h(\beta) < h(\lfloor\beta\rfloor) \leq h(\tau(Q^{\lfloor\beta\rfloor}))$.

We get by induction $P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \in \mathcal{SN}_{\mathcal{F}}$.

$$\begin{array}{c} (M_1 M_2)^\alpha \\ \downarrow \star \\ ((\cdots((x^\gamma\, A_1)^{\gamma_1}\, A_2)^{\gamma_2})\cdots A_n)^{\gamma_n}\, M_2)^\alpha \end{array}$$

$$\begin{array}{c} (M_1^*\, M_2^*)^\alpha \\ \downarrow \star \\ ((\cdots((N^\gamma\, A_1^*)^{\gamma_1}\, A_2^*)^{\gamma_2})\cdots A_n^*)^{\gamma_n}\, M_2^*)^\alpha \\ \downarrow \star \\ ((\lambda y.P)^\beta\, Q)^\alpha \\ \downarrow \\ P\{y := Q^{\lfloor\beta\rfloor}\}^{\lceil\beta\rceil\alpha} \end{array}$$

higher label