The cost of usage in the λ-calculus

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Plan

- the standardization theorem (with upper bounds)
- our result
- rigid and minimum prefixes (stability thm)
- Xi’s proof (with upper bounds)
- Xi’s proof revisited with live occurrences

.. joint work with Andrea Asperti (LICS 2013) ..
Shortest reductions

- non effective strategies

\[ M = F_m G_n \]

\[ G_n I G_n G_n \ldots G_n \]

\[ \Delta_n (lz) G_n G_n \ldots G_n \]

\[ \Delta_n z G_n G_n \ldots G_n \]

\[ F_m G'_n \]

\[ G'_n I G'_n G'_n \ldots G'_n \]

\[ (lz)(lz) \ldots (lz) G'_n G'_n \ldots G'_n \]

\[ F_m = \lambda x.x I x x \ldots x \]

\[ \Delta_n = \lambda x.x x \ldots x \]

\[ G_n = \lambda y.\Delta_n(yz) \]

\[ G'_n = \lambda y.(yz)(yz) \ldots (yz) \]
Standardization
Standard reductions (1/4)

• **Definition:** The following reduction is **standard**

\[ \rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N \]

iff for all \( i \) and \( j, i < j \), then \( R_j \) is not residual along \( \rho \) of some \( R_j' \) to the left of \( R_i \) in \( M_{i-1} \).

• **Definition:** The leftmost-outermost reduction is also called the **normal reduction**.
Standard reductions (2/4)
Standard reductions (3/4)

- **Standardization thm** [Curry 50]
  
  Let $M \rightarrow N$. Then $M \xrightarrow{\text{st}} N$.

- **Normalization corollary**
  
  Let $M \rightarrow nf$. Then $M \xrightarrow{\text{norm}} nf$.

Any reduction can be performed outside-in and left-to-right.
Standard reductions (4/4)

- **Head reduction corollary for values**

Let $M \rightarrow^* V$. Then $M \rightarrow_{\text{head}}^* \text{val}_{\text{min}}(M) \rightarrow^* V$
Our result

- **Upper-bound on standard reductions** [Hongwey Xi, 99]
  Let $\ell = |\rho|$ and $\rho : M \xrightarrow{\ast} N$. Then $|\rho_{st}| \leq |M|^{2^\ell}$
  where $\rho_{st} : M \xrightarrow{st} N$.

- **Upper-bound to normal forms** [Asperti-JJL, 13]
  Let $\ell = |\rho|$ and $\rho : M \xrightarrow{\ast} x$. Then $|\rho_{norm}| \leq \ell!$
  where $\rho_{norm} : M \xrightarrow{norm} x$.

We gain one exponential.
Standardization proofs

- **finite developments** \([\text{Klop, 80}]\)

Each reduction step is an empty reduction step.

Leftmost whose residual is contracted.
Standardization proofs

• **finite developments** [Gonthier–Melliès–JJL,92]
  tricky axiomatic proof

• **head normal forms** [Mitschke,80]

• **initial proof and statement** [Curry&Feys,70]
  correct statement, but proof ?
Standard reductions (4+/4)

- **Standardization thm** [JJL 77]
  Let $\rho : M \xrightarrow{\star} N$. $\exists! \rho_{st}. M \xrightarrow{\star}_{st} N$
  and $\rho_{st} \preceq \rho$.

  ![Diagram](image)

  Standard reduction is canonical representative in permutation class.

- **\(\lambda\)-standardization** [Church 36]
  Standard reduction is longest in its equivalence class.
Rigid prefixes: stability and multiplicity of variables
Stability (1/2)

• Definition [rigid prefix] Any rigid prefix $A$ of $M$ is any prefix of $M$ where never the left of an application can reduce to an abstraction.

$$M = \Omega(\lambda x.x(lx))(\lambda l.lx)$$

$$(\lambda x.x_{\_})_{\_}$ rigid prefix of $M$

$$(\lambda x.x_{\_})(\_ lx)$$ not rigid prefix of $M$

( rigid prefixes are finite prefixes of Berarducci trees)

• Definition $M$ produces $A$ if $M \overset{\rightarrow}{\longrightarrow} N$ and $A$ is rigid prefix of $N$. 

$$\Omega = (\lambda x.xx)(\lambda x.xx)$$

$I = \lambda x.x$
Stability (2/2)

• **Theorem** [stability] For any rigid prefix $A$ produced by $M$, there is a unique minimal prefix $[M]_A$ of $M$ producing $A$.

• **Fact** [monotony] Let $M$ produce $A$ rigid and $M \rightarrow N$. Then $N$ produces $A$. 
Slow consumption (1/2)

- **Lemma 1** [slow consumption] Let $M$ produce $A$ rigid and $M \rightarrow N$. Then $|\down{N}_A| \geq |\down{M}_A| - 2$.

  i.e. $\down{|M}_A|_{\oplus} \leq 1 + \down{|N}_A|_{\oplus}$ where $|P|_{\oplus}$ is the applicative size of $P$ (its number of application nodes).

- **Corollary** Let $\rho : M \rightarrow N$ and $A$ be rigid prefix of $N$. Then $\down{|M}_A|_{\oplus} \leq |\rho| + |A|_{\oplus}$. 
Slow consumption (2/2)

at most 2 nodes erased
Multiplicity of variables

• **Definition**  Let $M$ produce $A$ rigid. An occurrence of $x$ is live for $A$ if it belongs to $[M]_A$.

Let $m_A(x)$ be the number of live occurrences of $x$ in $M$. We pose $m_A(R) = m_A(x)$ when $R = (\lambda x.M)N$.

• **Lemma 2**  [upper bound on live multiplicity] Let $\rho : M \xrightarrow{*} N$ and $A$ rigid prefix of $N$. Then $m_A(x) \leq |\rho| + |A|_\emptyset + 1$ for any variable $x$ in $M$. 

Standardization
Xi’s proof of standardization (1/2)

- **Lemma** [reordering of head redexes] $H$ is residual of $H'$. Then

  $$M \xrightarrow[\rho_{\text{st}}]{} N = \lambda \bar{x}.(\lambda x.V)W \bar{N}$$

  $$M' \xrightarrow[\rho'_{\text{st}}]{} N' = \lambda \bar{x}.V\{x := W\} \bar{N}$$

  with $|\rho'| \leq \lceil 1, m(H) \rceil . |\rho|$

**Proof** Easy since $M = \lambda \bar{x}.(\lambda x.T)U \bar{M}$ and $\rho = \rho_T \rho_U \rho_1 \cdots \rho_n$. And $\rho'$ is disjoint intermix of $\rho_T$, several $\rho_U$, followed by $\rho_i$’s.

Thus $|\rho'| = |\rho_T| + m(H).|\rho_U| + \sum_i |\rho_i|$
Xi’s proof of standardization (2/2)

- **Corollary**

\[
M \xrightarrow{\rho} \text{st} \xrightarrow{\rho'} \text{st} \xrightarrow{R} N \xrightarrow{\text{st}} N'
\]

with \(|\rho'| \leq 1 + [1, m(R)].|\rho|\)

**Proof**

By induction on pair \((|\rho|, |M|)\). Cases on \(\rho R\) contracting head redex or not + previous lemma.
Xi’s proof of standardization (2/2)

• Theorem [standardization with upper bounds]

Let \( M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N \)

Then there is \( \rho \) standard from \( M \) to \( N \) such that

\[
|\rho| \leq (1 + \lceil 1, m(R_2) \rceil)(1 + \lceil 1, m(R_3) \rceil) \cdots (1 + \lceil 1, m(R_n) \rceil)
\]

Proof  By induction on the length \( n \) of reduction from \( M \) to \( N \).
Proof of our upper bound (1/2)

- **Theorem** [standardization with upper bounds]
  
  Let $M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$

  and $A$ be rigid prefix of $N$.

  Then there is $\rho$ standard from $M$ to $N'$ such that
  
  $|\rho| \leq (1 + \lceil 1, m_A(R_2) \rceil)(1 + \lceil 1, m_A(R_3) \rceil) \cdots (1 + \lceil 1, m_A(R_n) \rceil)$

  and $A$ is rigid prefix of $N'$. 
Proof of our upper bound (2/2)

- **Corollary 1** Let \( \rho : M \rightarrow^* N \) and \( A \) be rigid prefix of \( N \). Then there is \( \rho_{st} \) standard such that:

\[
|\rho_{st}| \leq \frac{(|\rho| + |A|_\circ)!}{(1 + |A|_\circ)!}
\]

**Proof** Simple calculation with lemma 2 and previous thm.

- **Corollary 2** Let \( \rho_{st} : M \rightarrow^* x \) be standard reduction. Then \( |\rho_{st}| \leq |\rho|! \) where \( \rho \) is shortest reduction from \( M \) to \( x \).
Conclusions
Conclusion

• terms are easy to grow in the $\lambda$-calculus

• but take time to consume terms

there is a need for sharing

• back to earth …. and higher-order functional languages