Small families
(at INRIA with Gérard and in the historical λ-calculus)

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sixty years is
31,557,600 minutes
and one minute is a
long time · · · · · · · · · · · ·
let us demonstrate
let us dem 1 nstrate
let us demonstrate
he passes the baccalauréat.
he writes his first program
and discovers functional programming.
he now is an attractive researcher
and starts a glorious academic life.
he passes 40 without care
still some hope
getting anxious
... and done!
Plan

1. strong normalisation
2. finite developments
3. redex families
4. generalised finite developments
5. conclusion
Strong normalisation
Typed $\lambda$-calculus

Theorem (Church-Rosser)

*The typed $\lambda$-calculus is confluent.*

and

Theorem (strong normalization)

*In typed $\lambda$-calculus, there are no infinite reductions.*

True at 1st order (Curry/Church), 2nd order (system F), ⋯ 60th order and even more ($F_\omega$, Coq).

Corollary

*The typed $\lambda$-calculus is a canonical system.*

The classical $\lambda$-calculus is confluent but provides infinite reductions: let $\Delta = \lambda x.xx$, then $\Omega = \Delta\Delta \rightarrow \Delta\Delta = \Omega$. 
Hyland-Wadsworth’s $D_\infty$-like $\lambda$-calculus

The idea is that $f_{n+1}(x) = (f(x_n))_n$ for $f, x \in D_\infty$

For the $\lambda$-calculus:

labels $m, n, p \geq 0$

expressions $M, N ::= x^n | (MN)^n | (\lambda x. M)^n$

$\beta$-conversion $((\lambda x. M)^{n+1} N)^p \rightarrow M\{x := N_{[n]}\}_{[n][p]}$

projection $x^n_{[m]} = x^p$

substitution $x^n\{x := P\} = P_{[n]}$

$(MN)^n_{[m]} = (MN)^p$

$(MN)^n\{x := P\} = (M\{x := P\}N\{x := P\})^n$

$(\lambda x. M)^n_{[m]} = (\lambda x. M)^p$

$(\lambda y. M)^n\{x := P\} = (\lambda y. M\{x := P\})^n$

where $p = [m, n]$

Notice that $((\lambda x.x^{58})^0 y^{59})^{60}$ is in normal form.

Degree of $((\lambda x. M)^n N)^p$ is $n$. 
Hyland-Wadsworth’s $D_{\infty}$-like $\lambda$-calculus

Let $\Omega_n = (\Delta_n \Delta_n)^n$, $\Delta_n = (\lambda x.(x^{60} x^{60})^{60})^n$

\[\Omega_{60} = \lambda x \rightarrow \lambda x \rightarrow \lambda x \rightarrow \lambda x = \Omega_{59}\]

Then
\[\Omega_{60} \rightarrow \Omega_{59} \rightarrow \Omega_{58} \rightarrow \cdots \Omega_1 \rightarrow \Omega_0\] in normal form
Up-to-60

\[ \beta\text{-conversion} \quad ((\lambda x. M)^n N)^p \rightarrow M\{x := N_{[n+1]}\}_{[n+1][p]} \]
when \( n \leq 60 \)

\[ \begin{align*}
\text{elevation} & \quad x^{m}_{[n]} = x^{p} \\
(MN)^{m}_{[n]} &= (MN)^{p} \\
(\lambda x. M)^{m}_{[n]} &= (\lambda x. M)^{p} \\
\text{where } p &= \lceil m, n \rceil
\end{align*} \]

\[ \begin{align*}
\text{substitution} & \quad x^{n}\{x := P\} = P_{[n]} \\
(MN)^{n}\{x := P\} &= (M\{x := P\} N\{x := P\})^{n} \\
(\lambda y. M)^{n}\{x := P\} &= (\lambda y. M\{x := P\})^{n}
\end{align*} \]

Theorem (Church-Rosser+SN)

\( \text{Hyland-Wadsworth and up-to-60 calculi are canonical systems.} \)

Comes from associativity of min/max since \([m, [n, p]] = [[m, n], p]\), and residuals keep degrees.
Any reduction graph $\mathcal{R}(M)$ can be approximated by an increasing chain of reduction graphs $\mathcal{R}_0(M), \mathcal{R}_1(M), \ldots, \mathcal{R}_{58}(M), \mathcal{R}_{59}(M), \mathcal{R}_{60}(M), \ldots$ of canonical systems.
Finite developments
Finite developments

Reductions of a set $\mathcal{F}$ of redexes in $M$ are described by:

- putting 0 on degrees of redexes in $\mathcal{F}$,
- putting 60 on degrees of other redexes,
- applying the up-to-1 calculus.

Theorem (finite developments — lemma of parallel moves)

There are no infinite reductions of a set $\mathcal{F}$ of redexes in $M$. All developments end on same term.

Proof: obvious since up-to-1 is a canonical system.

Theorem (finite developments+ — the cube lemma)

The notion of residuals is consistent with finite developments.
Created redexes

- Let $M_0$ have all subterms labeled by 0, let $M_0 \rightarrow N$ and $R$ redex in $N$ of non-zero degree, then $R$ is new redex (or created redex).

- Let $M = (\lambda x.x)(\lambda x.x)y$
  Then $M_0 = (((\lambda x.x^0)^0(\lambda x.x^0)^0)^0 y^0)^0 \rightarrow ((\lambda x.x^0)^1 y^0)^0$

- Let $\Omega = (\lambda x.xx)(\lambda x.xx)$
  Then $\Omega_0 = (\Delta^0 \Delta^0)^0 \rightarrow (\Delta^1 \Delta^1)^1 \rightarrow (\Delta^2 \Delta^2)^2 \rightarrow \cdots (\Delta^{60} \Delta^{60})^{60} \rightarrow \cdots$
  where $\Delta^n = (\lambda x.(x^0 x^0)^0)^n$

- redexes created (degree 0), redexes created by redex(es) created (degree 1), ... chains of creations. [event structures of $\lambda$-calculus]
Redex families
Let redex $R$ create redex $S$.

All $R$ redexes are residuals of $R$ redex in initial $\Delta R$.

The $S$ created redexes are not all residuals of a unique $S$.

But the $S$ redexes are only connected by a zigzag of residuals.

Furthermore the $S$ redexes are created in a “same way” by residuals of a same $R$-redex.
Let $\rho$ be reduction $M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} \cdots \xrightarrow{R_{60}} \cdots \xrightarrow{R_n} M_n$ and let $R$ be a redex in $M_n$.

**Definition**

We write $\langle \rho, R \rangle$ when $R$ is a redex in the final term of $\rho$. We say $R$ has history $\rho$.

The historical redexes $\langle \rho, R \rangle$ and $\langle \sigma, S \rangle$ are in a same family if connected by previous zigzag.

Histories are considered up to permutation equivalence $\sim$ on reductions.
The family equivalence between historical redexes is the symmetric, transitive, reflexive closure or the residual relation.

\[ \langle \rho_1, S \rangle \simeq \langle \rho_2, S_1 \rangle \simeq \langle \rho_4, S \rangle \simeq \langle \rho_2, S_2 \rangle \simeq \langle \rho_3, S \rangle \]
The historical $\lambda$-calculus

families of redexes by naming scheme of the labeled $\lambda$-calculus.

letters $a, b, c$

labels $\alpha, \beta, \gamma := a | \alpha[\beta] \gamma | \alpha[\beta] \gamma$

expressions $M, N := x^\alpha | (MN)^\alpha | (\lambda x. M)^\alpha$

$(h(\alpha) \leq 60)$ $\beta$-conversion

$((\lambda x. M)^\alpha N)^\beta \rightarrow \beta \cdot \lfloor \alpha \rfloor \cdot M\{x := \lfloor \alpha \rfloor \cdot N\}$

concat $\alpha \cdot x^\beta = x^{\alpha \beta}$

substitution $x^\alpha\{x := P\} = \alpha \cdot P$

$\alpha \cdot (MN)^\beta = (MN)^{\alpha \beta}$

$(MN)^\alpha\{x := P\} = (M\{x := P\}N\{x := P\})^\alpha$

$\alpha \cdot (\lambda x. M)^\beta = (\lambda x. M)^{\alpha \beta}$

$(\lambda y. M)^\alpha\{x := P\} = (\lambda y. M\{x := P\})^\alpha$

Theorem (Church-Rosser+SN)

The labeled (60 bounded) labeled $\lambda$-calculus is a canonical systems.

Comes from associativity of concatenation since $\alpha(\beta \gamma) = (\alpha \beta) \gamma$. 
Generalized Finite Developments
Finite developments revisited

Theorem (generalised finite developments – square lemma)

There are no infinite reductions of a finite set $\mathcal{F}$ of families in the reduction graph of $M$. All development of $\mathcal{F}$ end on same term.

Proof: obvious since up-to-$N$ strongly normalises where $N$ is the maximum degree of redexes in $\mathcal{F}$. For instance $N = 60$.

Theorem (finite developments+ — the cube lemma)

The notion of residuals of reductions (and hence of redex families) is consistent with generalised finite developments.

Corollary

A $\lambda$-term is strongly normalizable iff he only can create a finite number of redex families.
Finite chains of families creation

Only 3 cases of redex creations:

1. \((\lambda x. \cdots xN \cdots)(\lambda y. M) \rightarrow \cdots (\lambda y. M)N' \cdots\)
2. \((\lambda x.(\lambda y. M)N)P \rightarrow (\lambda y. M')P\)
3. \((\lambda x.x)(\lambda y. M)N \rightarrow (\lambda y. M)N\)

In 1st-order typed \(\lambda\)-calculus:

1. \((\lambda x. \cdots xN \cdots)^{\alpha \rightarrow \beta}(\lambda y. M) \rightarrow \cdots (\lambda y. M)^{\alpha}N' \cdots\)
2. \((\lambda x.\lambda y. M)^{\alpha \rightarrow \beta}NP \rightarrow (\lambda y. M')^{\beta}P\)
3. \((\lambda x.x)^{\alpha \rightarrow \alpha}(\lambda y. M)N \rightarrow (\lambda y. M)^{\alpha}N\)

In Hyland-Wadsworth’s \(\lambda\)-calculus:

1. \((\lambda x. \cdots xN \cdots)^{n+1}(\lambda y. M) \rightarrow \cdots (\lambda y. M)^{n}N' \cdots\)
2. \(((\lambda x.\lambda y. M)^{n+1}N)P \rightarrow (\lambda y. M')^{n}P\)
3. \((\lambda x.x)^{n+1}(\lambda y. M)N \rightarrow (\lambda y. M)^{n}N\)

Same in up-to-60 \(\lambda\)-calculus . . . .
Conclusion
Conclusion

- no infinite chain of creations is equivalent to strong normalisation.

- redex families exist also in TRS and many other reduction systems. E.g. redo it as permutation equivalences were treated in the almost everywhere rejected paper [Huet, Lévy 80]

- redo SN without the Tait/Girard reductibility incomprehensible reductibility method (with candidates or not).

- causality in reduction systems correspond to dependency, and can be useful for information flow, security – integrity properties. [Tomasz Blanc 06]

- understand more of the λ-calculus to be able to treat “real complex systems”
Future work

OBJECTIVE