Reductions and Causality (III)

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http://pauillac.inria.fr/~levy/courses/tsinghua/reductions

Plan

- properties of equivalence by permutations
- beyond lambda-calculus
- prefix ordering
- properties of prefix ordering
- the lattice of reductions
- canonical reductions

Equivalence by permutations

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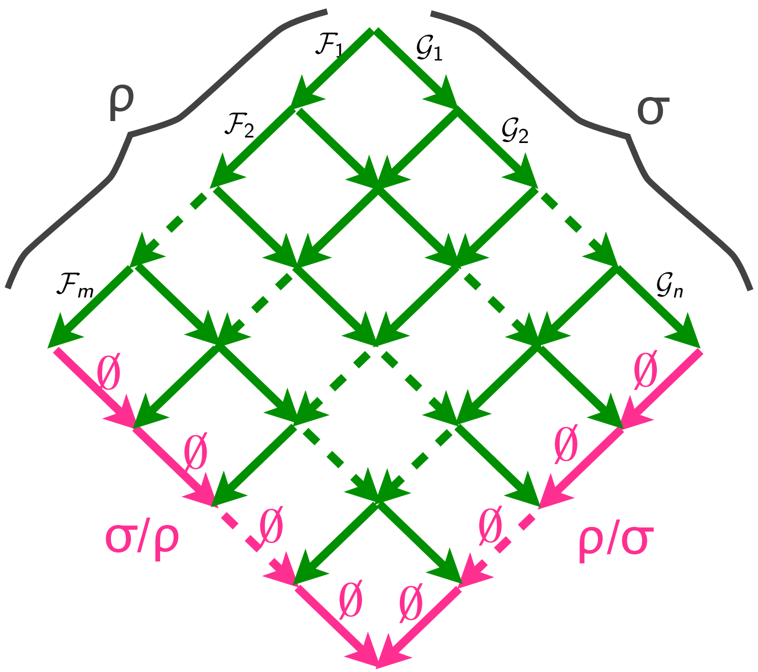
Equivalence by permutations

- **Definition:**

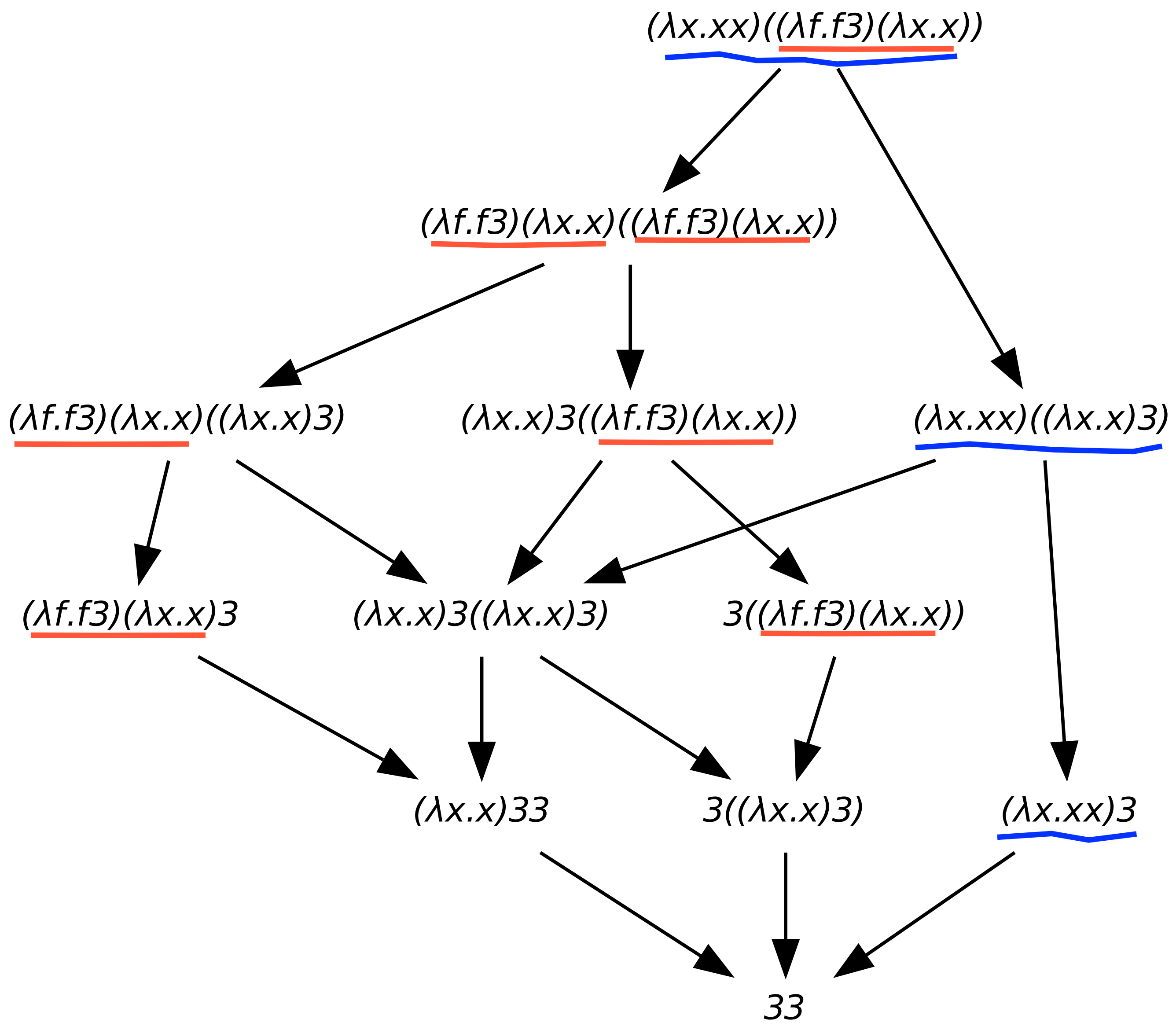
  Let ρ and σ be 2 coinitial reductions. Then ρ is equivalent to σ by permutations, $\rho \simeq \sigma$, iff:

$$\rho/\sigma = \emptyset^m \quad \text{and} \quad \sigma/\rho = \emptyset^n$$



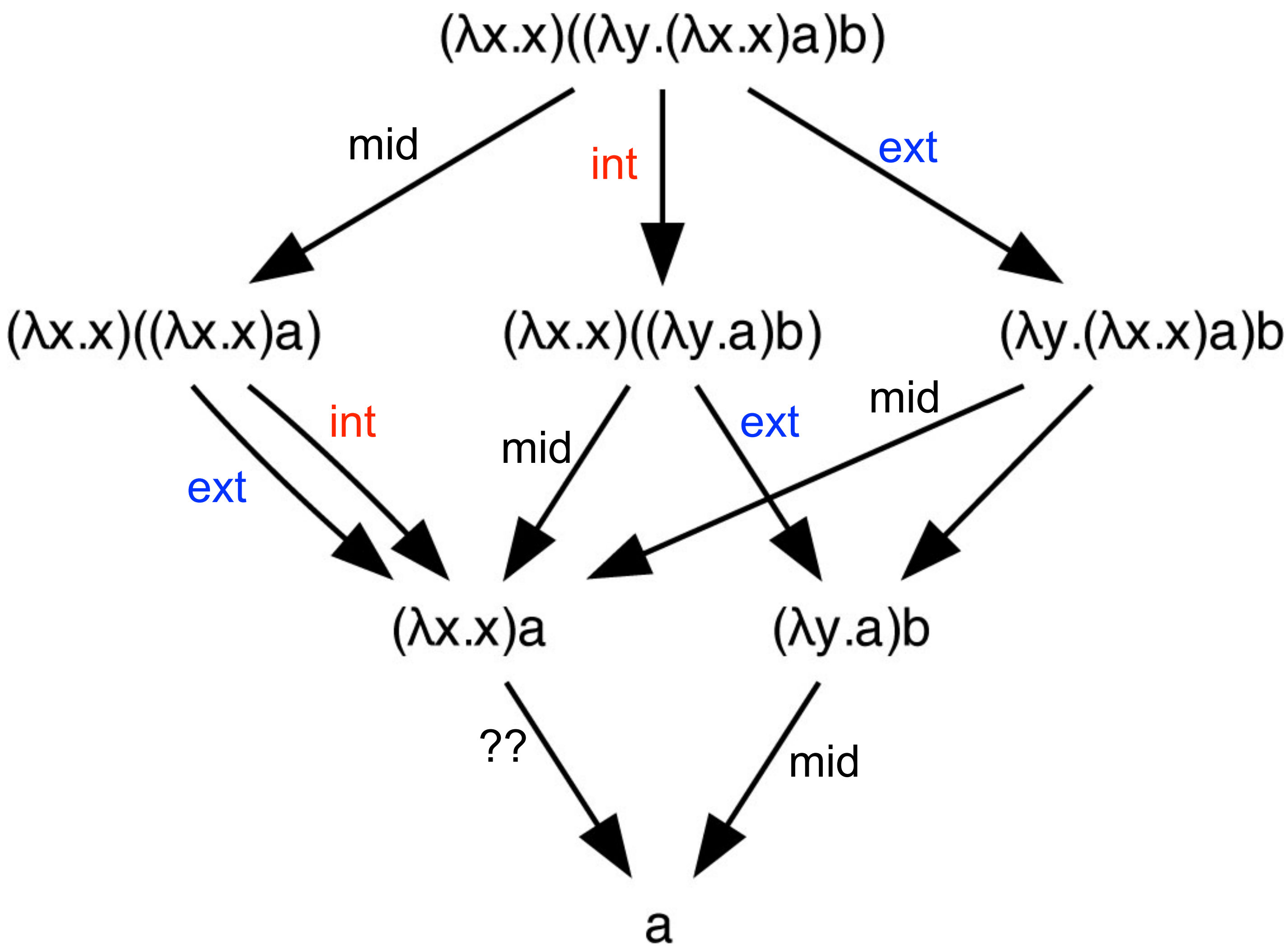
- Notice that $\rho \simeq \sigma$ means that ρ and σ are cofinal

Equivalence by permutations



- In this case, all coinitial&cofinal reductions are equivalent

Equivalence by permutations



- In this case, all coinitial&cofinal reductions are not equivalent

Equivalence by permutations

$$
\begin{tikzcd}
 & (\lambda x.x)((\lambda y.(\lambda x.x)a)b) \arrow[dl, "\text{mid}"'] \arrow[d, "\text{int}"] \arrow[dr, "\text{ext}"] & \\
(\lambda x.x)((\lambda x.x)a) \arrow[d] \arrow[dr] & (\lambda x.x)((\lambda y.a)b) \arrow[dl] \arrow[dr] & (\lambda y.(\lambda x.x)a)b \arrow[dl] \arrow[d] \\
(\lambda x.x)a \arrow[dr] & (\lambda x.x)a \arrow[d] & (\lambda y.a)b \arrow[dl] \\
 & a &
\end{tikzcd}
$$

- New reduction graph with equivalent reductions

Properties of perm. equivalence (1/3)

- **Proposition**

(a) $\rho \simeq \sigma$ iff $\forall \tau,\ \tau/\rho = \tau/\sigma$

(b) $\rho \sqcup \sigma \simeq \sigma \sqcup \rho$

(c) $\rho \simeq \sigma$ implies $\rho/\tau \simeq \sigma/\tau$

(d) $\rho \simeq \sigma$ iff $\tau;\rho \simeq \tau;\sigma$

(e) $\rho \simeq \sigma$ implies $\rho;\tau \simeq \sigma;\tau$

- **Proof**

(a) $\rho \simeq \sigma$ means $\sigma/\rho = \emptyset^n$. Therefore $\tau/\rho = (\tau/\rho)/(\sigma/\rho)$. That is $\tau/\rho = \tau/(\rho \sqcup \sigma)$. Similarly $\tau/\sigma = \tau/(\sigma \sqcup \rho)$. But cube lemma says $\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$. Therefore $\tau/\rho = \tau/\sigma$.

Conversely take $\tau = \rho$ and $\tau = \sigma$.

Properties of perm. equivalence (2/3)

- **Proposition**

  (a) $\rho \simeq \sigma$ iff $\forall \tau,\ \tau/\rho = \tau/\sigma$

  (b) $\rho \sqcup \sigma \simeq \sigma \sqcup \rho$

  (c) $\rho \simeq \sigma$ implies $\rho/\tau \simeq \sigma/\tau$

  (d) $\rho \simeq \sigma$ iff $\tau;\rho \simeq \tau;\sigma$

  (e) $\rho \simeq \sigma$ implies $\rho;\tau \simeq \sigma;\tau$

- **Proof**

  (b) (d) (e) Obvious by definition of residual.

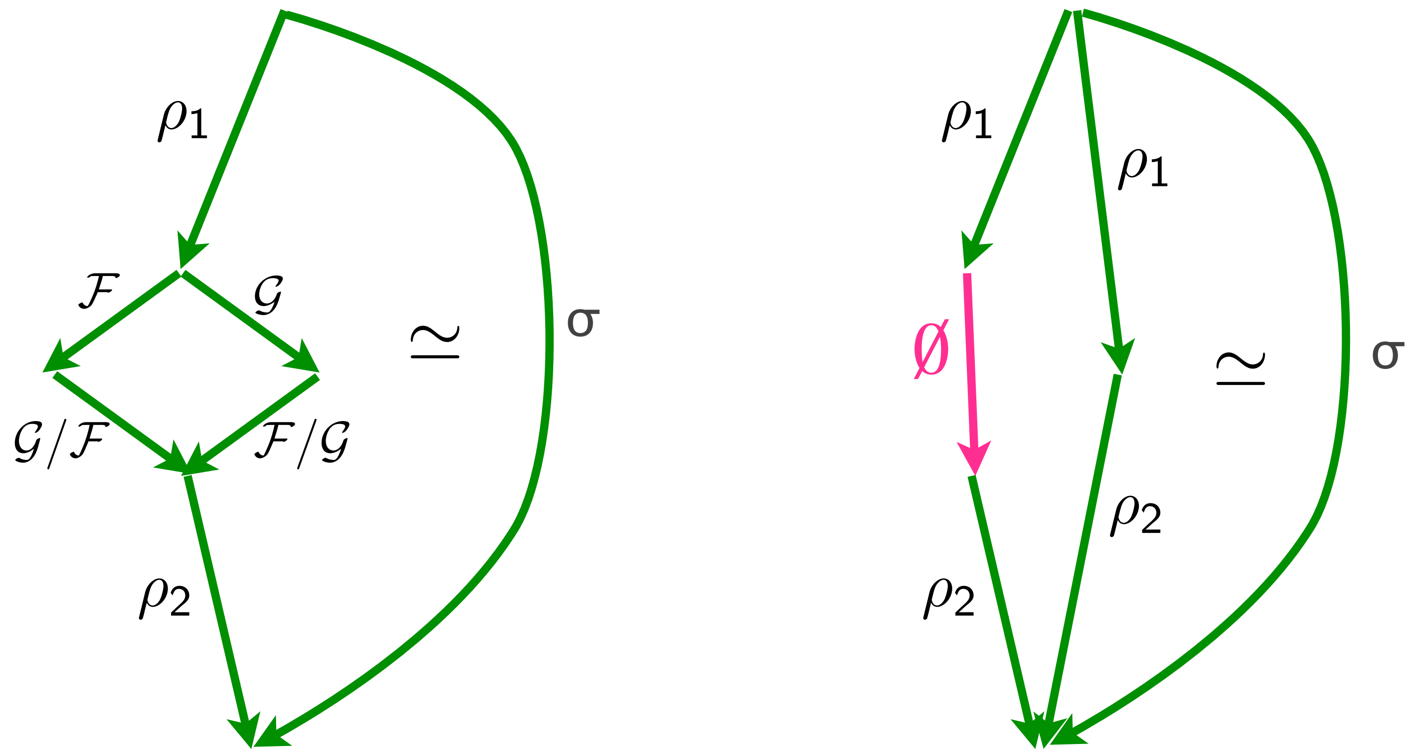
  (c) $(\rho/\tau)/(\sigma/\tau) = \rho/(\tau \sqcup \sigma) = \rho/(\sigma \sqcup \tau)$
  $= (\rho/\sigma)/(\tau/\sigma) = \emptyset^m$ by *(a)* and *(b)*.

Properties of perm. equivalence (3/3)

- **Proposition** $\simeq$ is the smallest congruence containing

$$\mathcal{F};(\mathcal{G}/\mathcal{F}) \simeq \mathcal{G};(\mathcal{F}/\mathcal{G})$$

$$0 \simeq \emptyset$$



Beyond λ-calculus

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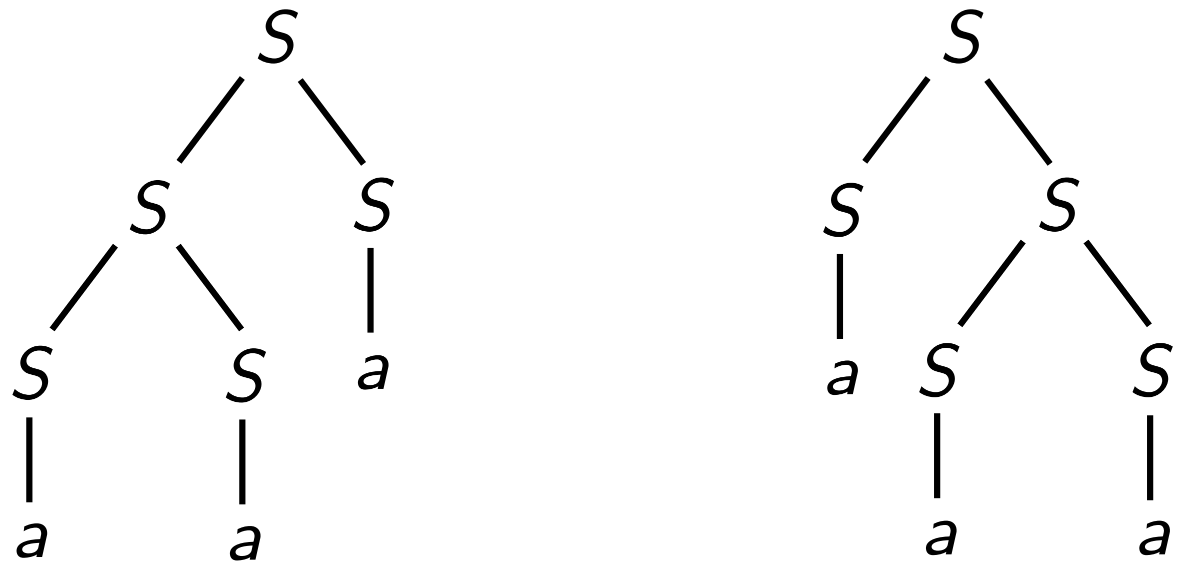
Context-free languages

- permutations of derivations in contex-free languages

$$S \to SS$$

$$S \to a$$



- each parse tree corresponds to an equivalence class of derivations

Term rewriting

- permutations of derivations are defined with critical pairs
- critical pairs make conflicts
- only 2nd definition of equivalence works for linear TRS

[Boudol, 1982]

Process algebras

- similar to TRS [Boudol-Castellani, 1988]

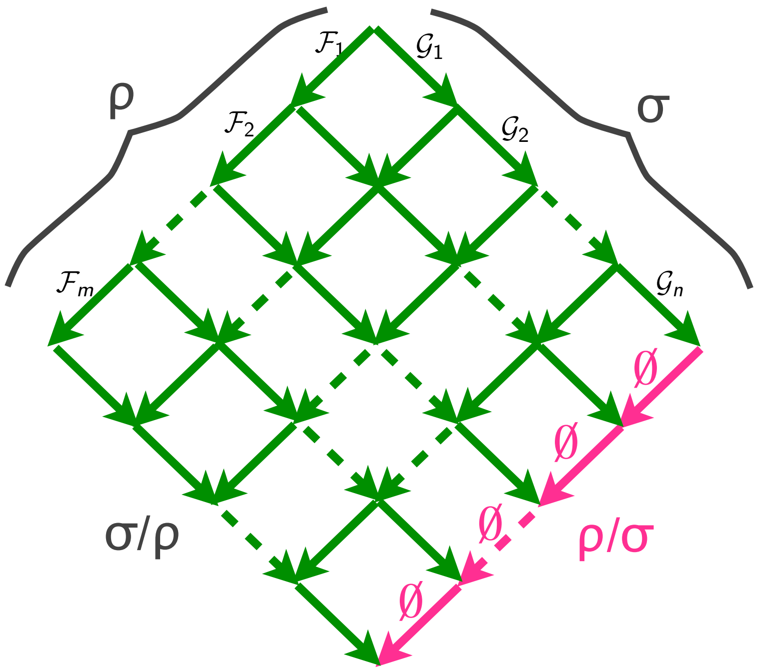
Weak memory models

- speculative computations [Boudol-Petri, 2009]

Prefix ordering

Prefix ordering (1/4)

- **Definition:**
  Let ρ and σ be 2 coinitial reductions. Then ρ is prefix of σ by permutations, $\rho \sqsubseteq \sigma$, iff $\rho/\sigma = \emptyset^m$



- Notice that $\rho \sqsubseteq \sigma$ means that $\rho \sqcup \sigma \simeq \sigma$

Properties of prefix ordering

- **Proposition**

(a) $\rho \sqsubseteq \sigma \sqsubseteq \rho$ iff $\rho \simeq \sigma$

(b) $\sqsubseteq$ is an ordering relation

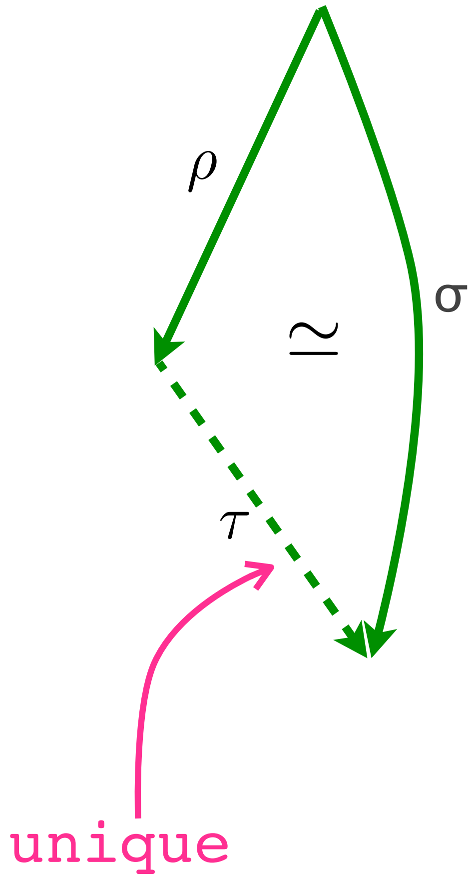
(c) $\rho \simeq \rho' \sqsubseteq \sigma' \simeq \sigma$ implies $\rho \sqsubseteq \sigma$

(d) $\rho \sqsubseteq \sigma$ iff $\tau ; \rho \sqsubseteq \tau ; \sigma$

(e) $\rho \sqsubseteq \sigma$ implies $\rho / \tau \sqsubseteq \sigma / \tau$

(f) $\rho \sqsubseteq \sigma$ iff $\exists \tau, \; \rho ; \tau \simeq \sigma$

(g) $\rho \sqsubseteq \sigma$ iff $\rho \sqcup \sigma \simeq \sigma$



Properties of prefix ordering

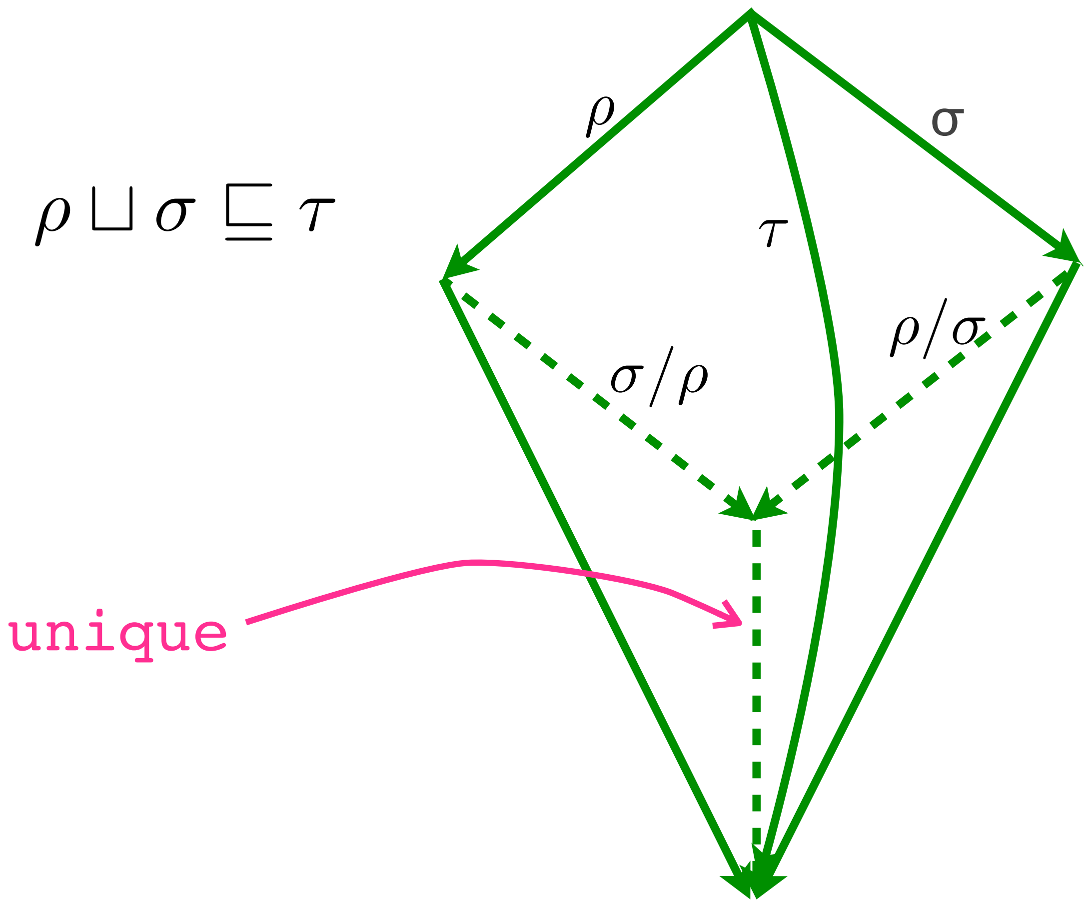
- **Proposition** [lattice of reductions]

  $\rho \sqsubseteq \rho \sqcup \sigma$

  $\sigma \sqsubseteq \rho \sqcup \sigma$

  $\rho \sqsubseteq \tau, \;\; \sigma \sqsubseteq \tau$ implies $\rho \sqcup \sigma \sqsubseteq \tau$

also named a *push-out*



Properties of prefix ordering

- **Proposition** `[lattice of reductions]`

  $\rho \sqsubseteq \rho \sqcup \sigma$

  $\sigma \sqsubseteq \rho \sqcup \sigma$

  $\rho \sqsubseteq \tau$, $\sigma \sqsubseteq \tau$ implies $\rho \sqcup \sigma \sqsubseteq \tau$

- **Proof** First two, already proved.

  Let $\rho \sqsubseteq \tau$, $\sigma \sqsubseteq \tau$. Then

  $(\rho \sqcup \sigma)/\tau$

  $= (\rho/\tau); ((\sigma/\rho)/(\tau/\rho))$

  $= \emptyset^m; \sigma/(\rho \sqcup \tau)$

  $= \emptyset^m; \sigma/(\tau \sqcup \rho)$

  $= \emptyset^m; (\sigma/\tau)/ \ldots$

  $= \emptyset^m; \emptyset^n/ \ldots = \emptyset^m; \emptyset^n$

Standard reductions

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Standard reductions (1/6)

- When R is a single redex, we write freely $R/\mathcal{F}$ for $\{R\}/\mathcal{F}$ or $\mathcal{F}/R$ for $\mathcal{F}/\{R\}$.

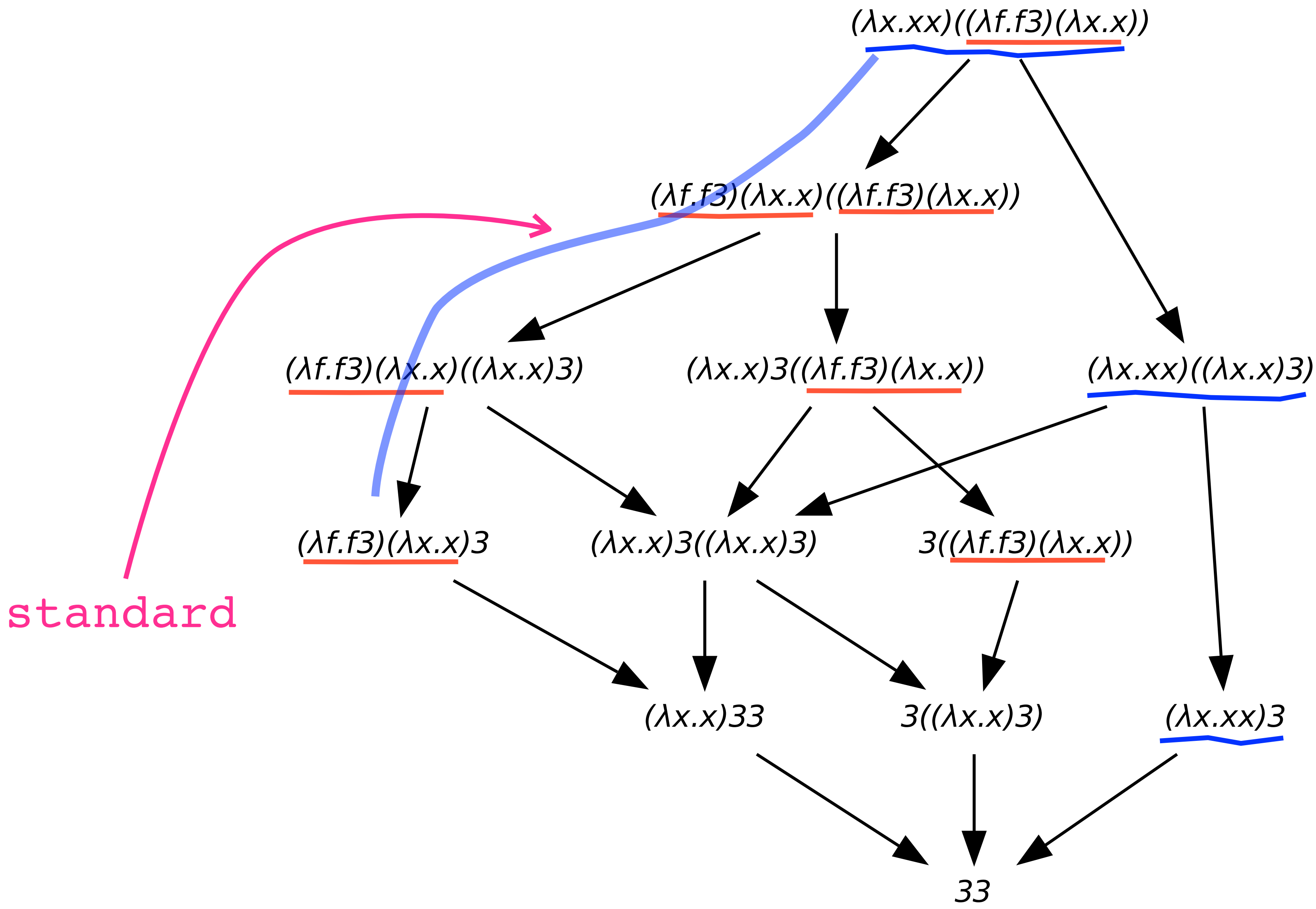
- **Proposition:**
  Let R be a redex to the left of $\mathcal{F}$. Then $R/\mathcal{F}$ is a singleton.

- **Definition:** The following reduction is **standard**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j to the left of R_i in M_{i-1}.

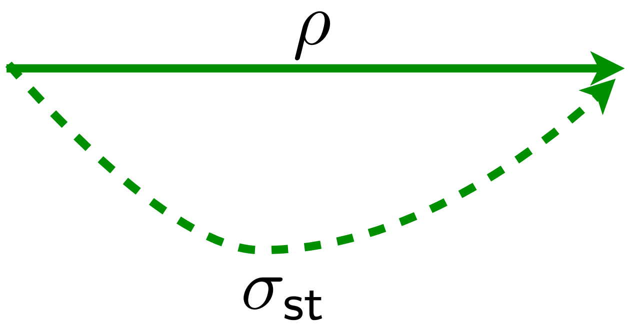
Standard reductions (2/6)



Standard reductions (3/6)

- **Standardization thm** [Curry 50]

  Let $M \xrightarrow{\star} N$. Then $M \xrightarrow[\text{st}]{\star} N$.



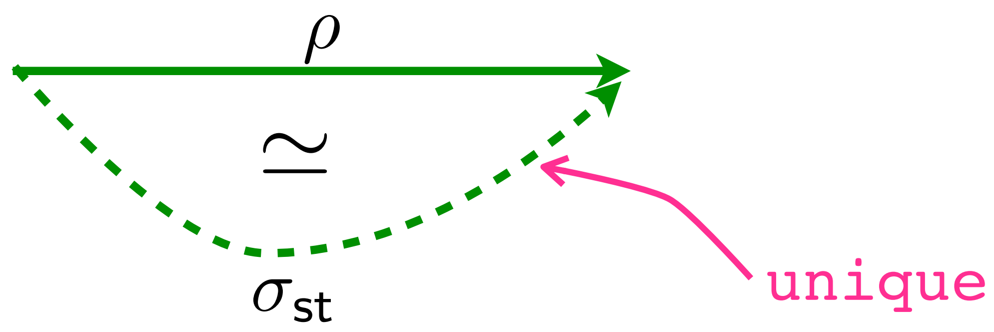
Any reduction can be performed outside-in and left-to-right.

Standard reductions (4/6)

- **Standardization thm +**

  Any ρ has a unique σ standard equivalent by permutations.

  $\forall \rho, \quad \exists!\, \sigma_{\mathsf{st}}, \quad \rho \simeq \sigma_{\mathsf{st}}$



Standard reductions are canonical representatives in their equivalence class by permutations.

Standard reductions (5/6)

- **Lemma** (left-to-right creation) [O'Donnell]
  Let R be redex to the left of redex S in M. Let $M \xrightarrow{\mathcal{S}} N$.
  If T' is redex in N to the left of the residual R' of R,
  T' is residual of a redex T in M.

$$M = \cdots \underbrace{((\lambda x. \cdots S \cdots)B)}_{R} \cdots \longrightarrow \cdots \underbrace{((\lambda x. \cdots S' \cdots)B)} \cdots = N$$

$$M = \cdots \underbrace{((\lambda x.A)(\cdots S \cdots))}_{R} \cdots \longrightarrow \cdots ((\lambda x.A)(\cdots S' \cdots)) \cdots = N$$

$$M = \cdots \underbrace{((\lambda x.A)B)}_{R} \cdots S \cdots \longrightarrow \cdots ((\lambda x.A)B) \cdots S' \cdots = N$$

One cannot create a new redex across another left one.

Standard reductions (6/6)

- **Lemma** If R to the left of R_1 and ρ is standard reduction starting with contracting R_1. Then $R/\rho \neq \emptyset$.

  Proof: application of previous lemma.

- **Proof of unicity of standard reduction in each equivalence class**

  Let ρ and σ be standard and $\rho \simeq \sigma$.
  They start with same reduction and differ at some point.
  Say that ρ is more to the left than σ. Then at that point redex R contracted by ρ has (unique) residual by σ.
  Therefore $\rho \not\simeq \sigma$.

Exercices

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Exercices

- Show all standard reductions in the 2 reduction graphs of beginning of this class.
- Show that all reductions to normal form are equivalent.
- Show that there is a single standard reduction to normal form. What is that reduction ?
- Find an example where there is no greatest lower bound of 2 reductions. (Hint: you should use K-terms)
- Show that there is inf-lattice of reductions in λI-calculus.
- Draw lattice of reductions of $\Delta\Delta$ ($\Delta = \lambda x.xx$).
- What are standard reductions in derivations of context-free languages ?