Reductions and Causality (II)

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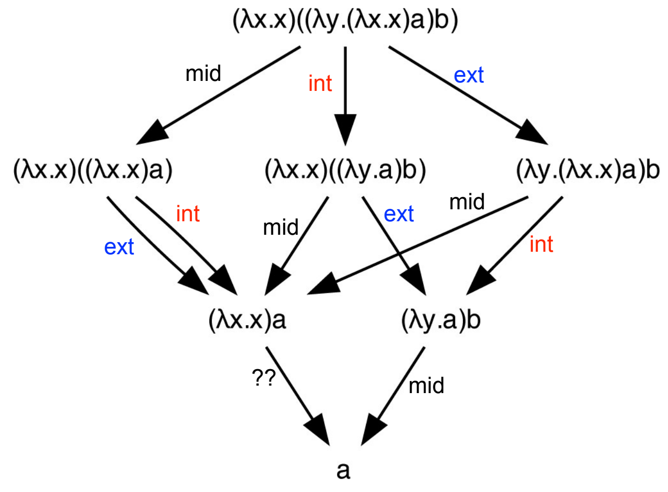
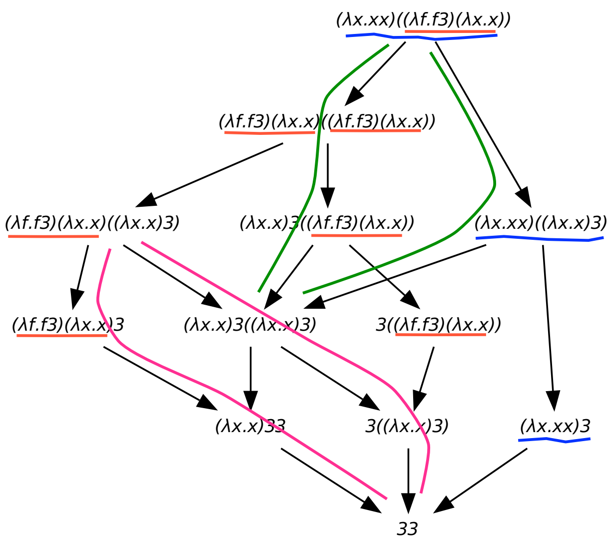
http://pauillac.inria.fr/~levy/courses/tsinghua/reductions

Exercice



Exercice



Plan

- parallel reduction steps
- cube lemma
- residuals of reductions
- equivalence by permutations
- beyond the λ-calculus

Parallel reduction steps

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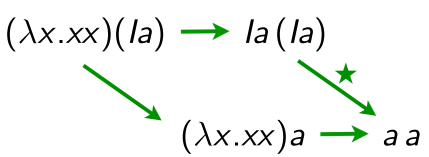
Parallel reductions (1/3)

- permutation of reductions has to cope with copies of redexes



- in fact, a parallel reduction $Ia(Ia) \mathrel{/\!\!/\!\!\!\!\rightarrow} aa$
- in λ-calculus, need to define parallel reductions for nested sets

Parallel reductions (2/3)

- the axiomatic way (à la Martin-Löf)

[Var Axiom] $x \mathrel{/\!\!/\!\!\!\!\rightarrow} x$ [Const Axiom] $c \mathrel{/\!\!/\!\!\!\!\rightarrow} c$

[App Rule] $\dfrac{M \mathrel{/\!\!/\!\!\!\!\rightarrow} M' \quad N \mathrel{/\!\!/\!\!\!\!\rightarrow} N'}{MN \mathrel{/\!\!/\!\!\!\!\rightarrow} M'N'}$ [Abs Rule] $\dfrac{M \mathrel{/\!\!/\!\!\!\!\rightarrow} M'}{\lambda x.M \mathrel{/\!\!/\!\!\!\!\rightarrow} \lambda x.M'}$

[//Beta Rule] $\dfrac{M \mathrel{/\!\!/\!\!\!\!\rightarrow} M' \quad N \mathrel{/\!\!/\!\!\!\!\rightarrow} N'}{(\lambda x.M)N \mathrel{/\!\!/\!\!\!\!\rightarrow} M'\{x := N'\}}$

- example:

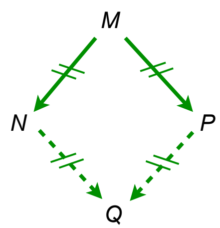
$$(\lambda x.Ix)(Iy) \mathrel{/\!\!/\!\!\!\!\rightarrow} (\lambda x.x)y$$
$$(\lambda x.(\lambda y.yy)x)(Ia) \mathrel{/\!\!/\!\!\!\!\rightarrow} Ia(Ia)$$
$$(\lambda x.(\lambda y.yy)x)(Ia) \mathrel{/\!\!/\!\!\!\!\rightarrow} (\lambda y.yy)a$$

- it's an *inside-out* parallel reduction

Parallel reductions (3/3)

- **Parallel moves lemma** `[Curry 50]`

If $M \mathrel{/\!\!/\!\!\!\!\rightarrow} N$ and $M \mathrel{/\!\!/\!\!\!\!\rightarrow} P$, then $N \mathrel{/\!\!/\!\!\!\!\rightarrow} Q$ and $P \mathrel{/\!\!/\!\!\!\!\rightarrow} Q$ for some Q.



lemma 1-1-1-1
(strong confluency)

- Enough to prove Church Rosser thm since $\rightarrow \subset \mathrel{/\!\!/\!\!\!\!\rightarrow} \subset \xrightarrow{*}$

`[Tait--Martin Löf 60?]`

// Reductions of set of redexes (1/4)

- Goal: parallel reduction of a given set of redexes

$$M, N ::= x \mid \lambda x.M \mid MN \mid (\lambda x.M)^a N$$
$$a, b, c, \ldots ::= \text{redex labels}$$

$$(\lambda x.M)N \longrightarrow M\{x := N\}$$
$$(\lambda x.M)^a N \longrightarrow M\{x := N\}$$

- Substitution as before with add-on:

$$((\lambda y.P)^a Q)\{x := N\} = (\lambda y.P\{x := N\})^a Q\{x := N\}$$

// Reductions of set of redexes (2/4)

- let $\mathcal{F}$ be a set of redex labels in M

[Var Axiom] $x \xrightarrow{\mathcal{F}} x$ [Const Axiom] $c \xrightarrow{\mathcal{F}} c$

[App Rule] $\dfrac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N'}{MN \xrightarrow{\mathcal{F}} M'N'}$ [Abs Rule] $\dfrac{M \xrightarrow{\mathcal{F}} M'}{\lambda x.M \xrightarrow{\mathcal{F}} \lambda x.M'}$

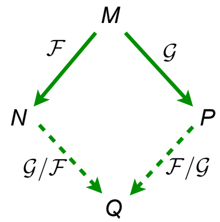
[//Beta Rule] $\dfrac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \in \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} M'\{x := N'\}}$ [Redex'] $\dfrac{M \xrightarrow{\mathcal{F}} M' \quad N \xrightarrow{\mathcal{F}} N' \quad a \notin \mathcal{F}}{(\lambda x.M)^a N \xrightarrow{\mathcal{F}} (\lambda x.M')^a N'}$

- let $\mathcal{F}$, $\mathcal{G}$ be set of redexes in M and let $M \xrightarrow{\mathcal{F}} N$, then the set $\mathcal{G}/\mathcal{F}$ of residuals of $\mathcal{G}$ by $\mathcal{F}$ is the set of $\mathcal{G}$ redexes in N.

// Reductions of set of redexes (3/4)

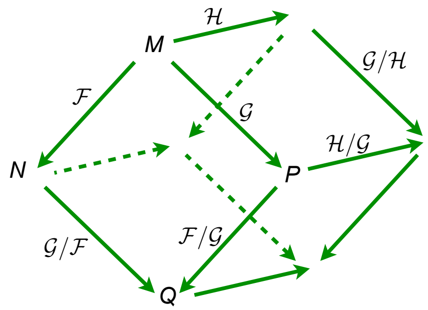
- **Parallel moves lemma+** [Curry 50]

If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.



// Reductions of set of redexes (4/4)

- **Parallel moves lemma++** [Curry 50] The Cube Lemma



- Then $(\mathcal{H}/\mathcal{F})/(\mathcal{G}/\mathcal{F}) = (\mathcal{H}/\mathcal{G})/(\mathcal{F}/\mathcal{G})$

Residuals of reductions

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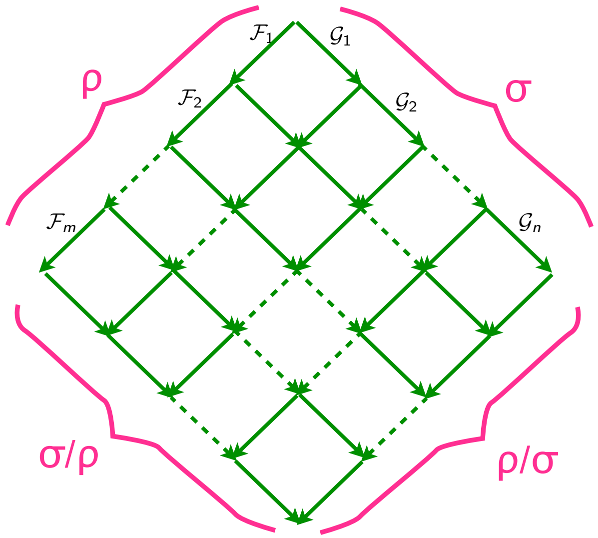
Residual of reductions (1/4)



Parallel reductions

- Redex occurences and labels
  - Let $||U|| = M$ where labels in U are erased (forgetful functor)
  - Then $M \xrightarrow{\mathcal{F}} N$ iff $U \xrightarrow{\mathcal{F}} N$ for some labeled U and $M = ||U||$
- Consider reductions where each step is parallel

$$\rho : M = M_0 \xrightarrow{\mathcal{F}_1} M_1 \xrightarrow{\mathcal{F}_2} M_2 \cdots \xrightarrow{\mathcal{F}_n} M_n = N$$

- We also write

$\rho = 0$ when $n = 0$

$\rho = \mathcal{F}_1; \mathcal{F}_2; \cdots \mathcal{F}_n$ when M clear from context

Residual of reductions (2/4)

- **Definition** [JJ 76]

$\rho/0 = \rho$

$\rho/(\sigma;\tau) = (\rho/\sigma)/\tau$

$(\rho;\sigma)/\tau = (\rho/\tau);(\sigma/(\tau/\rho))$

$\mathcal{F}/\mathcal{G}$ already defined

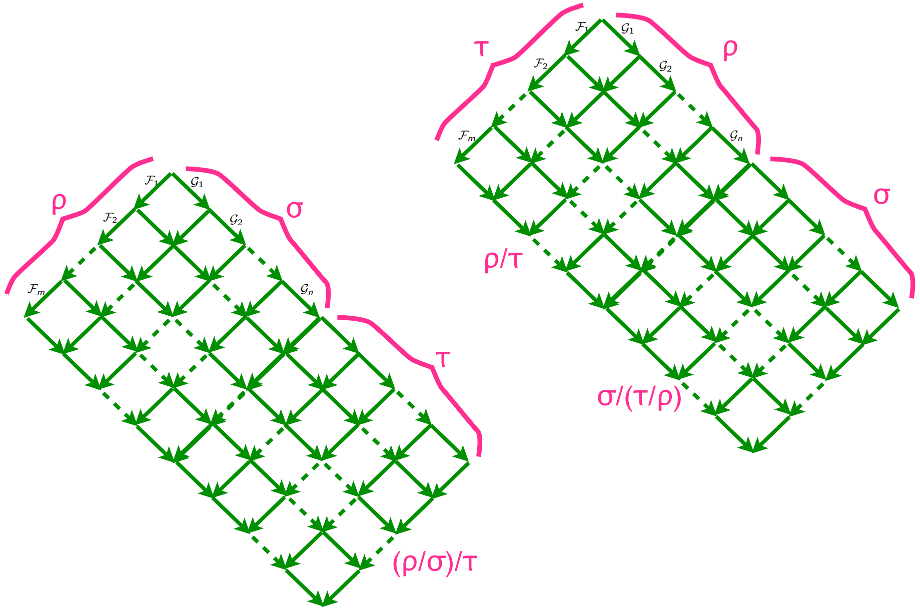
- **Notation**

$\rho \sqcup \sigma = \rho;(\sigma/\rho)$

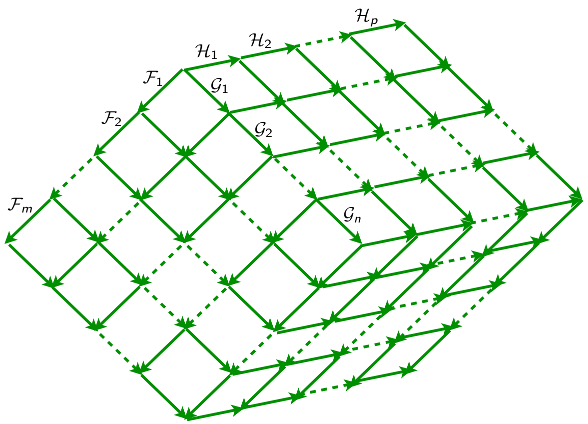
- **Proposition** [Parallel Moves +]:

$\rho \sqcup \sigma$ and $\sigma \sqcup \rho$ are cofinal

Residual of reductions (3/4)



Residual of reductions (4/4)



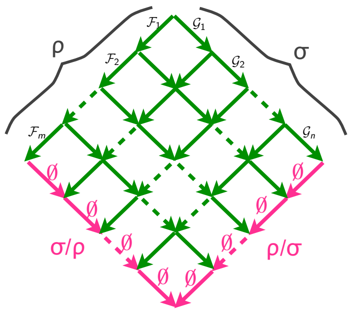
- **Proposition** [Cube Lemma ++]:

$$\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$$

Equivalence by permutations (1/4)

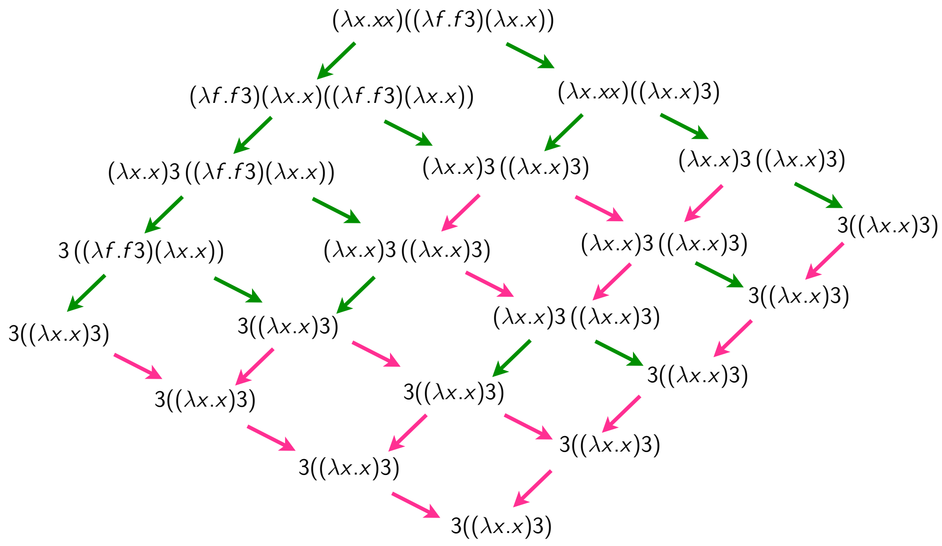
- **Definition:**
  Let ρ and σ be 2 coinitial reductions. Then ρ is equivalent to σ by permutations, $\rho \simeq \sigma$, iff:

$$\rho/\sigma = \emptyset^m \quad \text{and} \quad \sigma/\rho = \emptyset^n$$

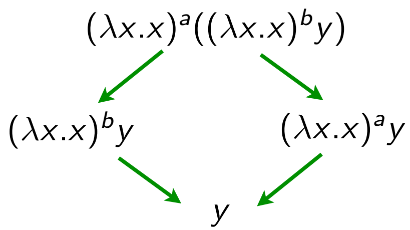


- Notice that $\rho \simeq \sigma$ means that ρ and σ are cofinal

Equivalence by permutations (2/4)



Equivalence by permutations (3/4)



$$\rho : M = I(Iy) \xrightarrow{R_a} Iy = N$$

$$\sigma : M = I(Iy) \xrightarrow{R_b} Iy = N$$

$$\rho \not\simeq \sigma$$

- Notice that $\rho \not\simeq \sigma$ while ρ and σ are coinitial and cofinal

Equivalence by permutations (4/4)

- Same with $0 \not\simeq \rho$ when $\rho : \Delta\Delta \longrightarrow \Delta\Delta$
  $\Delta = \lambda x.xx$

- **Exercice 1**: Give other examples of non-equivalent reductions between same terms

- **Exercice 2**: Show following equalities

$$\rho/0 = \rho \qquad \emptyset^n/\rho = \emptyset^n$$
$$0/\rho = 0 \qquad 0 \simeq \emptyset^n$$
$$\rho/\emptyset^n = \rho \qquad \rho/\rho = \emptyset^n$$

- **Exercice 3**: Show that $\simeq$ is an equivalence relation.

Properties of equivalent reductions

- **Proposition**

$$\rho \simeq \sigma \quad \text{iff} \quad \forall \tau,\ \tau/\rho = \tau/\sigma$$
$$\rho \simeq \sigma \text{ implies } \rho/\tau \simeq \sigma/\tau$$
$$\rho \simeq \sigma \quad \text{iff} \quad \tau;\rho \simeq \tau;\sigma$$
$$\rho \simeq \sigma \quad \text{implies} \quad \rho;\tau \simeq \sigma;\tau$$
$$\rho \sqcup \sigma \simeq \sigma \sqcup \rho$$

- **Proof**

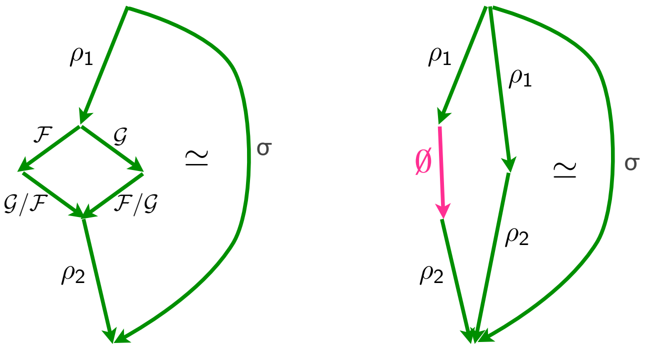
As $\rho \simeq \sigma$, one has $\sigma/\rho = \emptyset^n$. Therefore $\tau/\rho = (\tau/\rho)/(\sigma/\rho)$. That is $\tau/\rho = \tau/(\rho \sqcup \sigma)$. Similarly as $\sigma \simeq \rho$, one gets $\tau/\sigma = \tau/(\sigma \sqcup \rho)$. But cube lemma says $\tau/(\rho \sqcup \sigma) = \tau/(\sigma \sqcup \rho)$. Therefore $\tau/\rho = \tau/\sigma$.

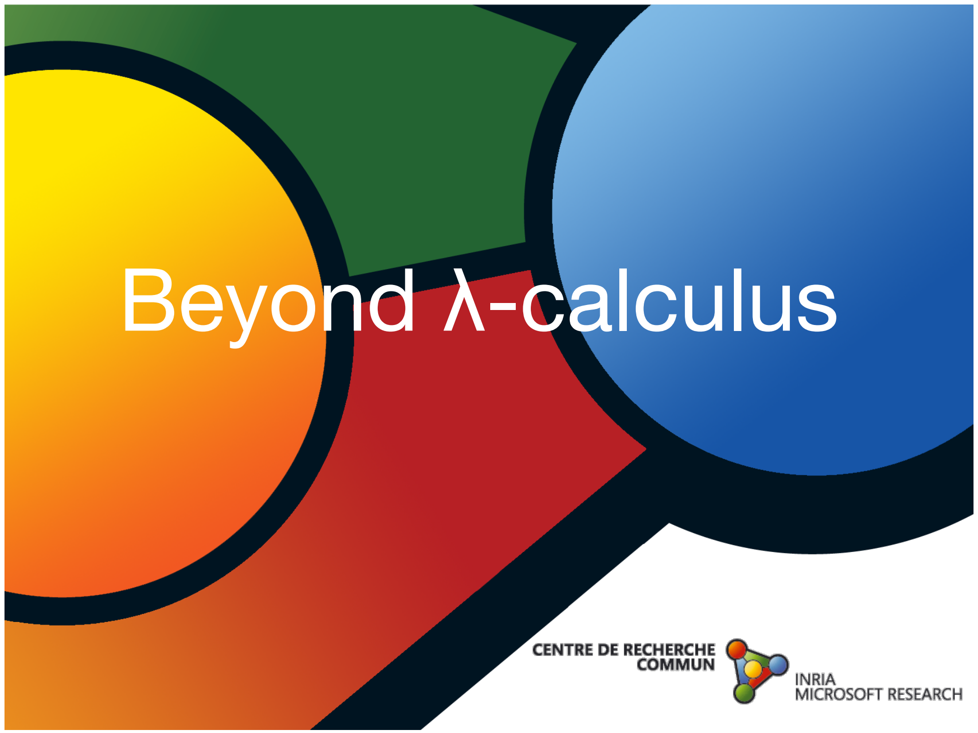
Properties of equivalent reductions

- **Proposition** $\simeq$ is the smallest congruence containing

  $\mathcal{F};(\mathcal{G}/\mathcal{F}) \simeq \mathcal{G};(\mathcal{F}/\mathcal{G})$

  $0 \simeq \emptyset$



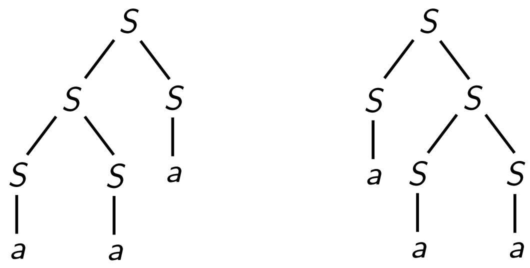


Context-free languages

- permutations of derivations in contex-free languages

$S \to SS$

$S \to a$



- each parse tree corresponds to an equivalence class

Term rewriting

- permutations of derivations are defined with critical pairs
- critical pairs make conflicts
- only 2nd definition of equivalence works `[Boudol, 1982]`

Process algebras

- similar to TRS `[Boudol-Castellani, 1982]`

Exercices

- **Exercice 4**: Complete all proofs of propositions
- **Exercice 5**: Show equivalent reductions in