Lambda-Calculus (III-6)

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Bohm tree semantics - reminders

- **Theorem [continuity]** For all $b \in \mathcal{N}$ such that $b \sqsubseteq C[M]$, then $b \sqsubseteq C[a]$ for some $a \in \mathcal{N}$ such that $a \sqsubseteq M$.
- **Theorem [monotony]** $M \sqsubseteq N$ implies $C[M] \sqsubseteq C[N]$
- **Theorem [λ-theory]** $M \equiv N$ implies $C[M] \equiv C[N]$

Proofs: easy consequences of previous proofs.

Plan

- Bohm trees -- reminders
- Morris extensional equivalence
- Bohm trees and η-rule
- Observational equivalences
- Relation with Scott's models

Approximations and η-rule

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Bohm tree and η-rule

- Functional extensionality has not been considered since we can have:

  $MP \equiv NP$ for all P, but $M \not\equiv N$.

  (Take $M = x$ and $N = \lambda y.xy$)

- We need take η-rule into account ! How to mix η-rule and Bohm tree construction ?
- We take for granted that $\xrightarrow{*}_\eta$ and $\xrightarrow{*}_{\beta,\eta}$ are confluent.
  Moreover $\xrightarrow{*}_\eta$ strongly normalizes.
- The prefix ordering between approximants must be extended. For instance:

  $$\lambda x.y\Omega \leq \lambda x.yx =_\eta y$$

  $$x\Omega =_\eta \lambda y.x\Omega y \leq \lambda y.xyy$$

Finite approximants

- We consider the set $\mathcal{N}^e$ of η-normal forms of finite approximants with following relation:

  $$a \leq^e b \quad \text{iff} \quad a\,(\leq \cup =_\eta)^*\, b$$

- **Lemma :** We have following commutation properties:

  $$\begin{array}{ccc} \rightarrow_\eta \leq & \subset & \leq \rightarrow_\eta \\ \leq \leftarrow_\eta & \subset & \leftarrow_\eta \leq \\ \rightarrow_\eta \leftarrow_\eta & \subset & \leftarrow_\eta \rightarrow_\eta \end{array}$$

- **Corollary:** $a \leq^e b$ iff $a \xleftarrow{*}_\eta a' \leq b' \xrightarrow{*}_\eta b$
- **Examples:**

  $$\lambda y.x\Omega \;\leq\; \lambda y.xy \xrightarrow{*}_\eta x$$
  $$x\Omega \xleftarrow{*}_\eta \lambda y.x\Omega y \;\leq\; \lambda y.xyy$$

Extensional Bohm trees

- **Definition :** Let $\omega^e(M)$ be the η-normal form of $\omega(M)$.
- **Definition :** The extensional Bohm tree $\mathrm{BT}^e(M)$ of M is defined by:

  $$\mathrm{BT}^e(M) = \{a \in \mathcal{N}^e \mid a \leq^e \omega^e(N),\; M \xrightarrow{*} N\}$$

- **Definition :** Extensional Bohm tree semantics

  $M \sqsubseteq^e N$ iff $\mathrm{BT}^e(M) \subset \mathrm{BT}^e(N)$
  $M \equiv^e N$ iff $\mathrm{BT}^e(M) = \mathrm{BT}^e(N)$

Extensional Bohm trees

- **Theorem :** $\sqsubseteq^e$ is a monotonic semantics and $\equiv^e$ forms a λ-theory.
- **Theorem** [Hyland, 1975]**:**

  $M \sqsubseteq^e N$ iff for all $C[\,]$, $C[M] \xrightarrow{*}$ n.f. implies $C[N] \xrightarrow{*}$ n.f.

J.Morris extensional equivalence
MIT-1968

Closed Bohm trees

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Finite Bohm trees revisited

- There are two ways of completing finite Bohm trees.

  1- standard completion by ideals (what we did)

  2- completion with closed ideals (does not add new limit point)

- We define the closure of directed sets as being the set with its already existing limit in $\mathcal{N}^e$

$$\mathrm{BT}^e_c(M) = \mathrm{cl}(\mathrm{BT}^e_c(M))$$

- We therefore have:

$$I \equiv^e_{\mathrm{cl}} J \quad \text{where}$$

$$J = Y(\lambda fxy.x(fy))$$

- and normal forms are no longer isolated points.

Finite Bohm trees revisited

- We just added η-rule to the standard Bohm tree construction with completion by ideals.
- However, we forgot slight difficulty: now, thanks to η-rule, finite Bohm tree now dominates an infinite number of other finite Bohm trees. See:

$$\begin{array}{ll}
a_\infty = x & I_\infty = \{a \in \mathcal{N}^e \mid a \leq^e x\} \\
 & \bigcup_{n=0}^{\infty} I_n \\
a_3 = \lambda x_1.x(\lambda x_2.x_1(\lambda x_3.x_2\Omega)) & I_3 = \{a \in \mathcal{N}^e \mid a \leq^e a_3\} \\
a_2 = \lambda x_1.x(\lambda x_2.x_1\Omega) & I_2 = \{a \in \mathcal{N}^e \mid a \leq^e a_2\} \\
a_1 = \lambda x_1.x\Omega & I_1 = \{a \in \mathcal{N}^e \mid a \leq^e a_1\} \\
a_0 = \Omega & I_0 = \{\Omega\}
\end{array}$$

new point
added by ideal completion

Finite Bohm trees revisited

- The equality between I and J is not so unnatural since one may emulate infinite η-expansion with the β-rule:

$$\begin{array}{rcl}
I = \lambda x.x & \leftarrow_\eta & \lambda xx_1.xx_1 \\
 & \leftarrow_\eta & \lambda xx_1.x(\lambda x_2.x_1x_2) \\
 & \leftarrow_\eta & \lambda xx_1.x(\lambda x_2.x_1(\lambda x_3.x_2x_3)) \\
 & \leftarrow_\eta & \cdots \\
J = Y(\lambda fxx_1.x(fx_1)) & =_\beta & \lambda xx_1.x(Jx_1) \\
 & =_\beta & \lambda xx_1.x(\lambda x_2.x_1(Jx_2)) \\
 & =_\beta & \lambda xx_1.x(\lambda x_2.x_1(\lambda x_3.x_2(Jx_3))) \\
 & =_\beta & \cdots
\end{array}$$

- same phenomenon as for these two versions of identity on natural numbers:

$$\begin{array}{rcl}
I(n) & \Leftarrow & n \\
J(n) & \Leftarrow & \mathtt{if}\ n = 0\ \mathtt{then}\ 1\ \mathtt{else}\ 1 + J(n-1)
\end{array}$$

Bohm trees and Scott's models

- We have following correspondances:

  1- $M \sqsubseteq^e_c N$ iff $M \sqsubseteq_{D_\infty} N$ (Scott's model)

  2- $M \sqsubseteq N$ iff $M \sqsubseteq_{T^\omega} N$ (Plotkin's model)

- One can also show that:

  3- $M \sqsubseteq^{e+} N$ iff $M \sqsubseteq_{P_\omega} N$ (Scott's model)

  where $\sqsubseteq^{e+}$ is Bohm tree construction from $\xleftarrow{*}_\eta \le$ ordering on $\mathcal{N}^e$.

- finally, one may order Bohm trees with symmetrical:

  4- $M \sqsubseteq^{e-} N$

  where $\sqsubseteq^{e-}$ is Bohm tree construction from $\le \xrightarrow{*}_\eta$ ordering on $\mathcal{N}^e$.

Homeworks

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Observational equivalences

- To conclude we have the following results:

  1- $M \sqsubseteq^e_c N$ iff for all contexts $C[\,]$, $C[M] \xrightarrow{*}$ hnf implies $C[N] \xrightarrow{*}$ hnf

  2- $M \sqsubseteq^e N$ iff for all contexts $C[\,]$, $C[M] \xrightarrow{*}$ nf implies $C[N] \xrightarrow{*}$ nf

- Making observational equivalences with other Bohm tree semantics is more difficult since one has to fight with η-equality. Take for instance $\lambda x.xx$ and $\lambda x.x(\lambda y.xy)$

Exercices

1- Show that if M has no hnf, then M is totally undefined.

2- Show $\Omega M \equiv \Omega$ and $\lambda x.\Omega \equiv \Omega$. Show that $M \longrightarrow_\omega N$, then $M \equiv N$.

3- Find M and N such that $MP \equiv NP$ for all P, but $M \not\equiv N$. (Meaning that $\equiv$ is not extensional)

4- Show $M \not\equiv \lambda x.Mx$ when $x \notin \mathrm{var}(M)$. What if $M \equiv \lambda x.M_1$?

5- Let $Y_0 = Y$, $Y_{n+1} = Y_n(\lambda xy.y(xy))$. Show that $Y \equiv Y_n$ for all n. However all Y_n are pairwise non interconvertible.

6 If $M \le P$ and $N \le P$ (M and N are prefix compatible), then $\mathrm{BT}(M \sqcap N) = \mathrm{BT}(M) \cap \mathrm{BT}(N)$. (Thus BT is stable in Berry's sense, 1978). What if not compatible ?

Exercices

7- `[Barendregt 1971]`

A closed expression M (i.e. $\mathrm{var}(M) = \emptyset$) is solvable iff:

$\forall P,\ \exists N_1, N_2, \ldots N_n$ such that $MN_1N_2 \cdots N_n =_\beta P$

(in short:

$\forall P,\ \exists \vec{N},\ M\vec{N} =_\beta P$)

Show that for every closed term M, the following are equivalent:

1. M has a hnf
2. $\exists \vec{N},\ M\vec{N}$ has a normal form
3. $\exists \vec{N},\ M\vec{N} =_\beta I$
4. M is solvable

8- `[Barendregt 1974]`

Show that, in the λI-calculus, a term M is solvable iff it has a normal form.

Exercices

9- Let $\mathcal{R}$ be a preorder on $\mathcal{N}$ (reflexive + transitive) compatible with its structure:

$a_1 \mathrel{\mathcal{R}} b_1, \ldots a_n \mathrel{\mathcal{R}} b_n$ implies $xa_1a_2 \cdots a_n$

$a \mathrel{\mathcal{R}} b$ implies $\lambda x.a \mathrel{\mathcal{R}} \lambda x.b$

Let $M \sqsubseteq_{\mathcal{R}} N$ iff $\forall a \in \mathrm{BT}(M),\ \exists b \in \mathrm{BT}(N),\ a \mathrel{\mathcal{R}} b$

Show that when M is a closed term, one has:

$\forall \vec{P},\ M\vec{P} \sqsubseteq_{\mathcal{R}} N\vec{P}$ iff $\forall C[\,],\ C[M] \sqsubseteq_{\mathcal{R}} C[N]$

10- (cont'd 1) Let $M \mathrel{\mathcal{R}} N$ be "if M has a normal form, then N has a normal form"

Give examples of M and N such that $M \sqsubseteq_{\mathcal{R}} N$ but $M \not\sqsubseteq N$.

11- (cont'd 2) Let $M \mathrel{\mathcal{R}} N$ be "if M has a hnf, then N has a hnf"

Give examples of M and N such that $M \sqsubseteq_{\mathcal{R}} N$ but $M \not\sqsubseteq N$.

12- (cont'd 2) Let $M \mathrel{\mathcal{R}} N$ be "if M has a hnf, then N has a similar hnf"

Give examples of M and N such that $M \sqsubseteq_{\mathcal{R}} N$ but $M \not\sqsubseteq N$. (Hint: consider $M = \lambda x.xx$ and $N = \lambda x.x(\lambda y.xy)$) [Compare with `Hyland 1975`])