Lambda-Calculus (III-5)

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Head normal forms

- A term is in **head normal form** (hnf) iff it has the following form:

$$\lambda x_1 x_2 \cdots x_m . x M_1 M_2 \cdots M_n \text{ with } m \geq 0 \text{ and } n \geq 0$$

head variable

(x may be free or bound by one of the x_i)

- A term not in **head normal form** is of following form:

$$\lambda x_1 x_2 \cdots x_m . (\lambda x . M) N N_1 N_2 \cdots N_n$$

head redex

- Head normal forms appeared in **Wadsworth's** phD [1973].

Plan

- Finite Bohm trees
- Infinite Bohm trees
- Monotony and Continuity theorems
- Inside-out completeness
- Generalized Finite Developments
- Another labeled calculus

Bohm trees

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Bohm trees

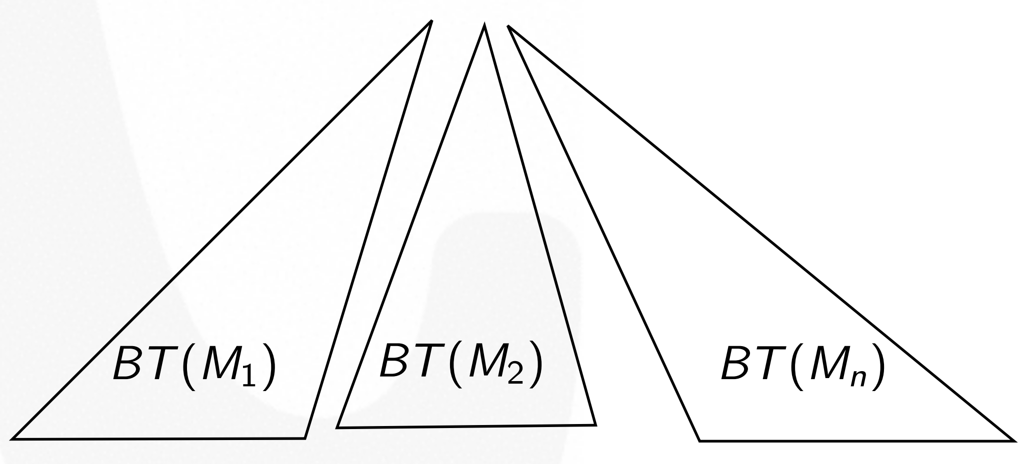
- Intuitively:

If M has no hnf

$$BT(M) = \Omega$$

If $M \xrightarrow{*} \lambda x_1 x_2 \cdots x_m . x \ M_1 \ M_2 \ \cdots \ M_n$

$$BT(M) = \lambda x_1 x_2 \cdots x_m . x$$



Bohm trees

$$BT(\Delta\Delta) = \Omega$$

$$BT(Ix(Ix)(Ix)) = \begin{array}{ccc} & x & \\ / & & \backslash \\ x & & x \end{array}$$

$$BT(Ix(\Delta\Delta)(Ix)) = \begin{array}{ccc} & x & \\ / & & \backslash \\ \Omega & & x \end{array}$$

$$BT(Ix(Ix)(\Delta\Delta)) = \begin{array}{ccc} & x & \\ / & & \backslash \\ x & & \Omega \end{array}$$

$$BT(Y) = \begin{array}{c} \lambda f.f \\ | \\ f \\ | \\ f \\ | \\ \vdots \end{array} = BT(Y')$$

$$Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$$

$$Y' = (\lambda xy.y(xxy))(\lambda xy.y(xxy))$$

Finite Bohm trees

- A **finite approximant** is any member of the following set of terms:

$$
\begin{array}{lcl}
a, b & ::= & \Omega \\
 & | & \lambda x_1 x_2 \cdots x_m . x a_1 a_2 \cdots a_n \quad (m \geq 0, n \geq 0)
\end{array}
$$

- examples of finite approximants:

$$x\Omega\Omega$$

$$xx\Omega$$

$$x\Omega x$$

$$\lambda xy.xy(x\Omega)$$

$$\lambda xy.x(\lambda z.y\Omega)$$

- we call $\mathcal{N}$ the set of finite approximants

Finite Bohm trees

- Finite approximants can be ordered by following **prefix ordering**:

$$\Omega \leq a$$

$$a_1 \leq b_1,\ a_2 \leq b_2,\ \ldots\ a_n \leq b_n \text{ implies}$$

$$\lambda x_1 x_2 \cdots x_m . x a_1 a_2 \cdots a_n \leq \lambda x_1 x_2 \cdots x_m . x b_1 b_2 \cdots b_n$$

- examples:

$$x\Omega\Omega \leq xx\Omega$$

$$x\Omega\Omega \leq x\Omega x$$

$$\lambda xy.x\Omega \leq \lambda xy.xy$$

- thus $a \leq b$ iff several Ω's in a are replaced by finite approximants in b.

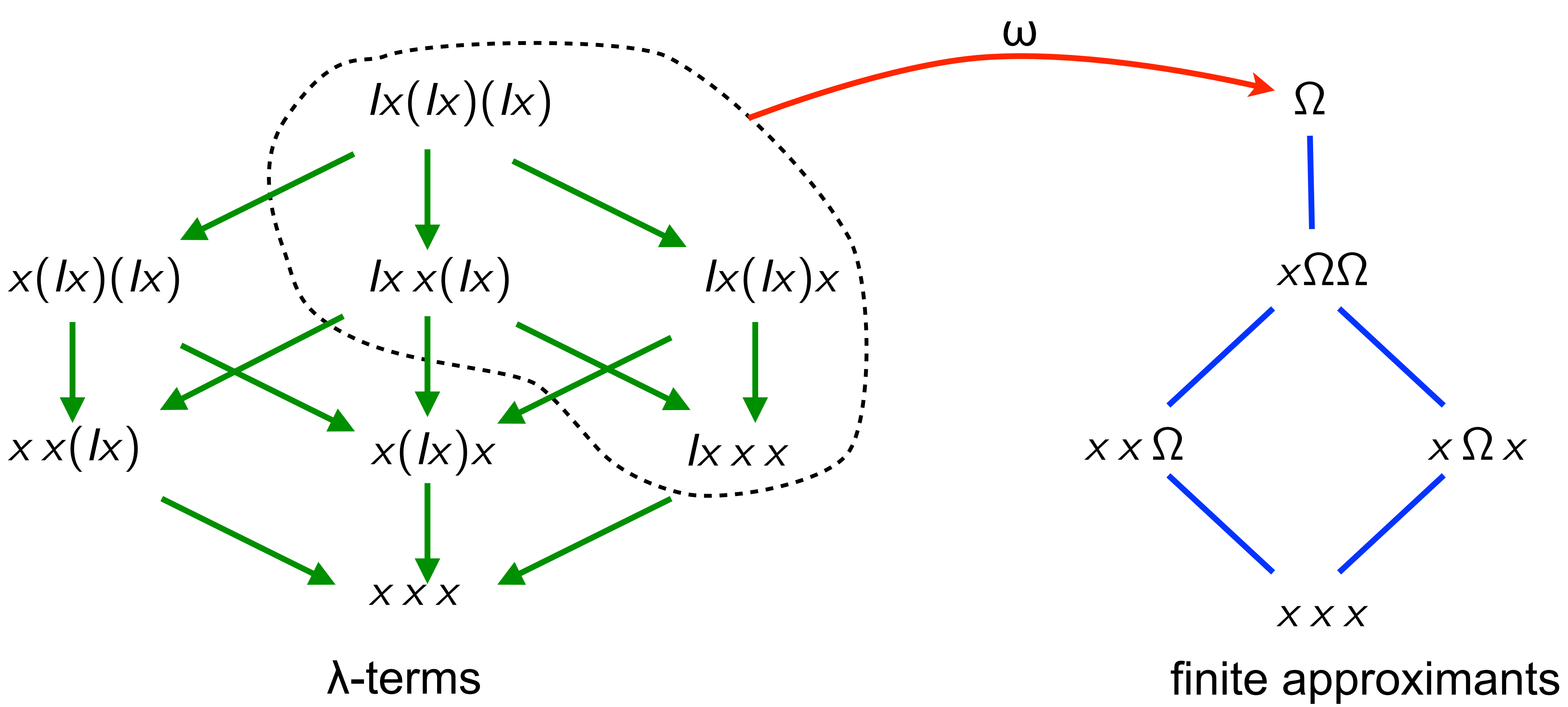
Finite Bohm trees

- $\omega(M)$ is **direct approximation of** *M*. It is obtained by replacing all redexes in *M* by constant Ω and applying exhaustively the two Ω-rules:

$$\Omega M \longrightarrow \Omega$$

$$\lambda x.\Omega \longrightarrow \Omega$$

- examples of direct approximation:



Finite Bohm trees

- **Lemma 1:**

  $\omega(M) = \Omega$ iff M is not in hnf.

  $\omega(\lambda x_1 x_2 \cdots x_m . x M_1 M_2 \cdots M_n) = \lambda x_1 x_2 \cdots x_m . x(\omega(M_1))(\omega(M_2)) \cdots (\omega(M_n))$

- **Lemma 2:** $M \longrightarrow N$ implies $\omega(M) \leq \omega(N)$

- **Lemma 3:** The set $\mathcal{N}$ of finite approximants is a conditional lattice with $\leq$.

- **Definition:** The set $\mathcal{A}(M)$ of direct approximants of M is defined as:

$$\mathcal{A}(M) = \{\omega(N) \mid M \xrightarrow{*} N\}$$

- **Lemma 4:** The set $\mathcal{A}(M)$ is a sublattice of $\mathcal{N}$ with same lub and glb.

  Proof: easy application of Church-Rosser + standardization.

Bohm trees

- **Definition:** The Bohm tree of M is the set of prefixes of its direct approximants:

$$\mathrm{BT}(M) = \{a \in \mathcal{N} \mid a \leq b,\ b \in \mathcal{A}(M)\}$$

- In the terminology of partial orders and lattices, Bohm trees are ideals. Meaning they are directed sets and closed downwards. Namely:

directed sets: $\forall a, b \in \mathrm{BT}(M), \exists c \in \mathrm{BT}(M), \quad a \leq c \ \wedge \ b \leq c.$

ideals: $\forall b \in \mathrm{BT}(M), \forall a \in \mathcal{N}, a \leq b \ \Rightarrow \ a \in \mathrm{BT}(M).$

- In fact, we made a completion by ideals. Take $\overline{\mathcal{N}} = \{A \mid A \subset \mathcal{N},\ A \text{ is an ideal}\}$

Then $\langle \mathcal{N}, \leq \rangle$ can be completed as $\langle \overline{\mathcal{N}}, \subset \rangle$.

- Thus Bohm trees may be infinite and they are defined by the set of all their finite prefixes.

Bohm trees

- **Examples:**

1- $\mathrm{BT}(\Delta\Delta) = \{\Omega\} = \mathrm{BT}(\Delta\Delta\Delta) = \mathrm{BT}(\Delta\Delta M)$

2- $\mathrm{BT}((\lambda x.xxx)(\lambda x.xxx)) = \mathrm{BT}(YK) = \{\Omega\}$

3- $\mathrm{BT}(M) = \{\Omega\}$ if M has no hnf

4- $\mathrm{BT}(I) = \{\Omega, I\}$

5- $\mathrm{BT}(K) = \{\Omega, K\}$

6- $\mathrm{BT}(Ix(Ix)(Ix)) = \{\Omega, x\Omega\Omega, xx\Omega, x\Omega x, xxx\}$

7- $\mathrm{BT}(Y) = \{\Omega, \lambda f.f\Omega, \lambda f.f(f\Omega), \ldots \lambda f.f^n(\Omega), \ldots\}$

8- $\mathrm{BT}(Y') = \{\Omega, \lambda f.f\Omega, \lambda f.f(f\Omega), \ldots \lambda f.f^n(\Omega), \ldots\}$

Bohm tree semantics

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Bohm tree semantics

- **Definition 1:** let the Bohm tree semantics be defined by:

$$M \equiv_{\mathrm{BT}} N \;\text{ iff }\; \mathrm{BT}(M) = \mathrm{BT}(N)$$

- **Definition 2:** we also consider Bohm tree ordering defined by:

$$M \sqsubseteq_{\mathrm{BT}} N \;\text{ iff }\; \mathrm{BT}(M) \subset \mathrm{BT}(N)$$

When clear from context, we just write $\equiv$ for $\equiv_{\mathrm{BT}}$ and $\sqsubseteq$ for $\sqsubseteq_{\mathrm{BT}}$.

- **New goal:** is Bohm tree semantics a (consistent) λ-theory ?

- We want to show that:

$$M \xrightarrow{\star} N \;\text{ implies }\; M \equiv N$$

$$M \sqsubseteq N \;\text{ implies }\; C[M] \sqsubseteq C[N]$$

Bohm tree semantics

- **Proposition 1:** $M \xrightarrow{\star} N$ implies $M \equiv N$

Proof: First $\mathrm{BT}(N) \subset \mathrm{BT}(M)$, since any approximant of N is one of M.
Conversely, take a in $\mathrm{BT}(M)$. We have $a \leq b = \omega(M')$ where $M \xrightarrow{\star} M'$.
By Church-Rosser, there is N' such that $M' \xrightarrow{\star} N'$ and $N \xrightarrow{\star} N'$. By lemma 1, we have $\omega(M') \leq \omega(N')$.
Therefore $a \leq \omega(N')$ and $a \in \mathrm{BT}(N)$.

Bohm tree semantics

- Let consider λ-calculus (all set of λ-terms) with extra constant Ω and corresponding prefix ordering, β-conversion and straitforward extension of Bohm tree semantics.

- **Lemmas:**

  1- $M \leq N$ implies $M \sqsubseteq N$

  2- $a \in \mathrm{BT}(M)$ implies $a \sqsubseteq \mathrm{BT}(C[M])$

Proof:

1- First notice that if $M \leq N$ and $M \xrightarrow{*} M'$, then $N \xrightarrow{*} N'$ with $M' \leq N'$ for some N'. Therefore if a be in $BT(M)$, there is M' such that $M \xrightarrow{*} M'$ and $a \leq \omega(M')$. So there is N' such that $M' \xrightarrow{*} N'$ and $N \xrightarrow{*} N'$. So $a \leq \omega(M') \leq \omega(N')$ by lemma 2. Thus a is also in $\mathrm{BT}(N)$.

2- Let a be in $\mathrm{BT}(M)$. Consider b in $\mathrm{BT}(a)$. This means $b \leq a$.

We have $a \leq \omega(P)$ with $C[M] \xrightarrow{*} P$. Thus $a \leq P$. By previous lemmas, we have $a \sqsubseteq P \equiv C[M]$. Therefore $a \sqsubseteq C[M]$.

Bohm tree semantics

- Remember we considered completion $\langle\overline{\mathcal{N}}, \subset\rangle$ by ideals of $\langle\mathcal{N}, \leq\rangle$.
- Therefore we have an upper limit $\cup S$ of any directed subset S in $\overline{\mathcal{N}}$.
  (One has just to check that $\cup S$ is an ideal of $\mathcal{N}$)
- **Proposition 2:** $M \sqsubseteq N$ implies $C[M] \sqsubseteq C[N]$

  Proof: we already know by previous lemmas:

  $$\cup\{C[a] \mid a \in \mathrm{BT}(M)\} \ \subset\ \cup\{C[b] \mid b \in \mathrm{BT}(N)\} \ \subset\ \mathrm{BT}(C[N])$$

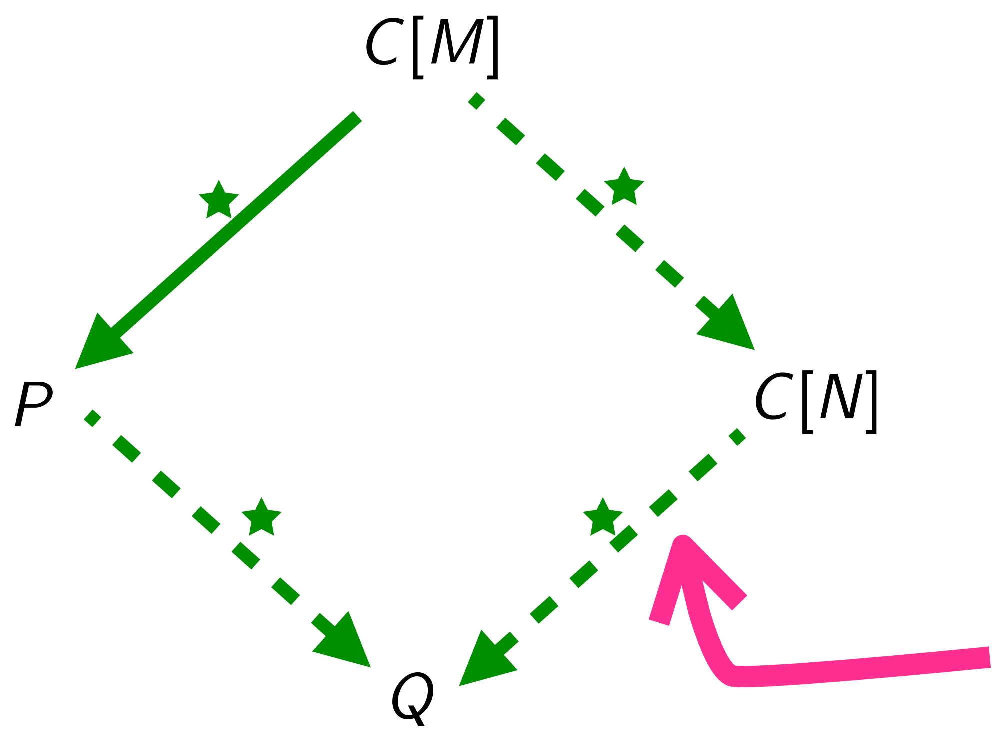
  Remains to show $\mathrm{BT}(C[M]) \subset \cup\{C[a] \mid a \in \mathrm{BT}(M)\}$!

  I.e. $\forall b \in \mathrm{BT}(C[M]),\ \exists a \in BT(M),\ \ b \in BT(C[a])$??

  I.e. continuity of context w.r.t Bohm tree semantics !!

Bohm tree semantics

- We want to show following property `[Welch, 1974]`



reduction not contracting residuals of redexes in *N*

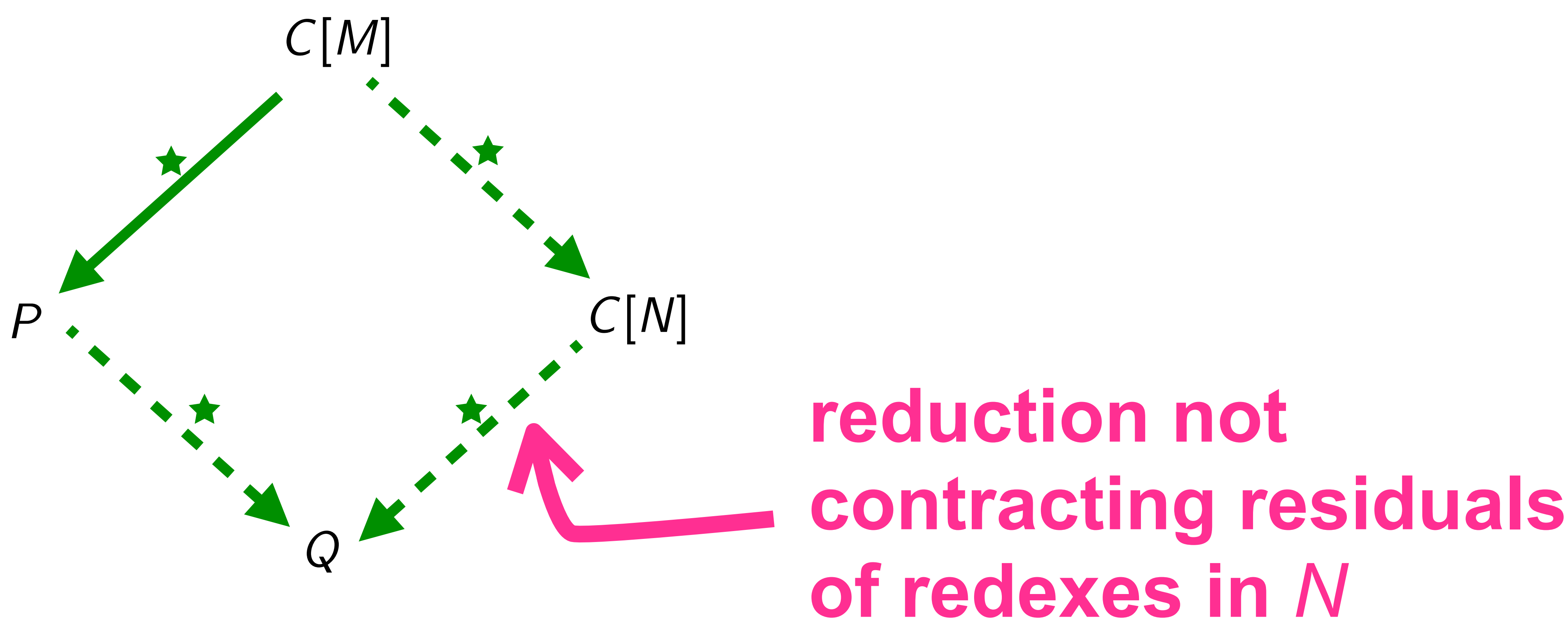
First one show that for any A and set of redexes $\mathcal{F}$ in A. If $A \longrightarrow A'$ without contracting a redex in $\mathcal{F}$, then $A\{\mathcal{F} := \Omega\} \longrightarrow A'\{\mathcal{F}' := \Omega\}$ where $\mathcal{F}'$ are the residuals of $\mathcal{F}$.

Then let $b \leq \omega(P)$. One has $b \leq \omega(P) \leq \omega(Q)$ and thus $b \sqsubseteq \omega(Q)$. Now let $\mathcal{F}'$ are residuals of the set $\mathcal{F}$ of redexes in N within $C[N]$, one has:

$\omega(Q) \sqsubseteq Q\{\mathcal{F}' := \Omega\}$ since $\omega(Q) \leq Q\{\mathcal{F}' := \Omega\}$,

Bohm tree semantics

- We want to show following property `[Welch, 1974]`



$Q\{\mathcal{F}' := \Omega\} \equiv C[N\{F := \Omega\}]$ since they are β-inconvertible,

$C[N\{\mathcal{F} := \Omega\}] \equiv C[a]$ since $C[N\{\mathcal{F} := \Omega\}] \xrightarrow{*}_{\omega} C[a]$.

Therefore $b \sqsubseteq C[a]$, meaning $b \in \mathrm{BT}(C[a])$ since a is finite.

Bohm tree semantics

- **Theorem [continuity]** For all $b \in \mathcal{N}$ such that $b \sqsubseteq C[M]$, then $b \sqsubseteq C[a]$ for some $a \in \mathcal{N}$ such that $a \sqsubseteq M$.

- **Theorem [monotony]** $M \sqsubseteq N$ implies $C[M] \sqsubseteq C[N]$

- **Theorem [λ-theory]** $M \equiv N$ implies $C[M] \equiv C[N]$

Proofs: easy consequences of previous proofs.

Exercices

1- Show that $M \sqsubseteq N$ for all N when M has no hnf.

2- [algebraicity] Show that $a \sqsubseteq M$ implies $a \in \mathrm{BT}(M)$ for any $a \in \mathcal{N}$.

3- Show that if M has a normal form and $M \sqsubseteq N$, then M and N have same normal form.

4- Show that if M has a hnf and $M \sqsubseteq N$, then M and N have similar hnfs.

5- Show that $Yf \equiv Yf^2$.

6- Show that $Y(f \circ g) \equiv f(Y(g \circ f))$

7- Show that any monotonic semantics $\sqsubseteq'$ such that $\Omega \sqsubseteq' M$ for any M also satisfies $\Omega M \equiv' \Omega$. How about $\lambda x.\Omega \equiv' \Omega$?

8- Show $Y \equiv Y'$ for any Y' such that $Y'f \equiv f(Y'f)$.

Generalized Finite Developments

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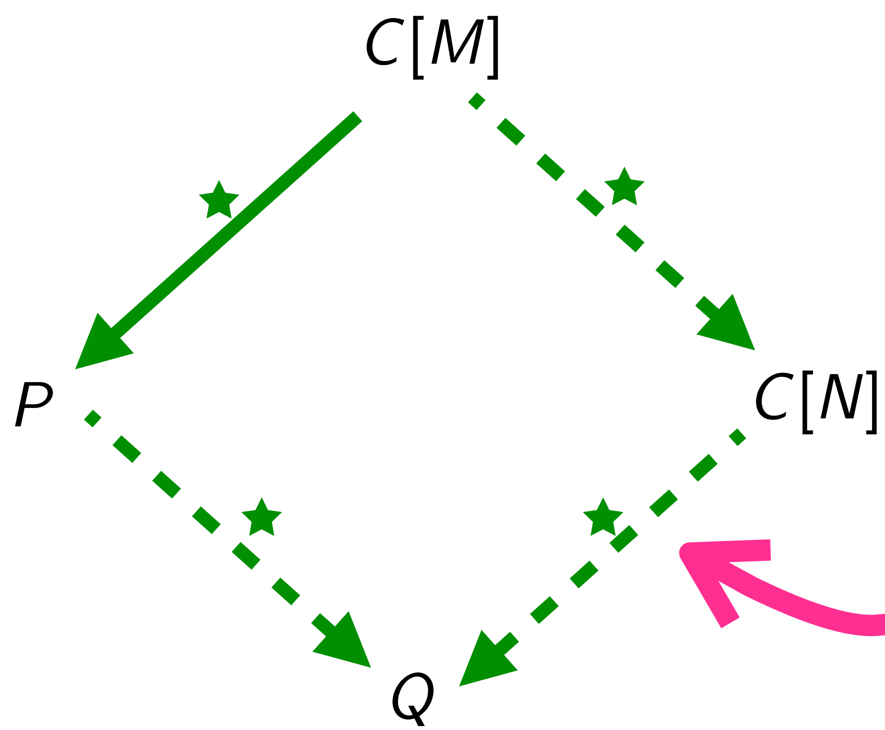
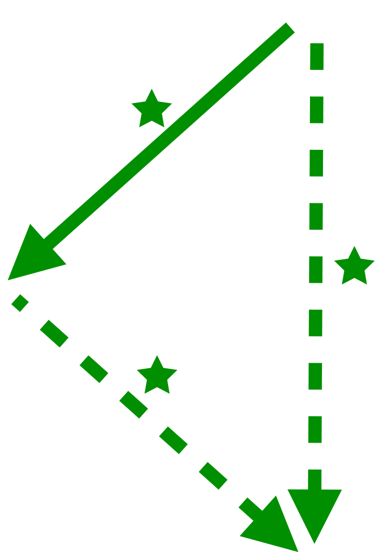
Inside-out reductions

- How to prove the following property `[Welch, 1974]`



reduction not contracting residuals of redexes in *N*

- It can be derived from following simpler property.



inside-out reduction

Inside-out reductions

- **Definition:**
  The reduction $M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$ is inside-out iff for all i, j $(0 < i < j \leq n)$, redex R_j is not a residual of redex R'_j inside R_i in M_{i-1}.

- How to prove it ? Intuitively one just have to reorder redexes contracted in any given reduction and get an inside-out reduction maybe getting further than initial reduction because of symmetries forced by the inside-out order.

- Another remark is that if M strongly normalizes, one has just to consider any innermost reduction until its normal form.

Another labeled calculus

- We add a natural number as exponent of any subterm.
- **Lambda calculus with indexes à la Scott-Wadsworth-Hyland**

M, N, P	$::=$	x^n	(variables)
	$\mid$	$(\lambda x.M)^n$	(M as function of x)
	$\mid$	$(M\ N)^n$	(M applied to N)

- **Labeled reduction**

$$((\lambda x.M)^{n+1}N)^p \longrightarrow M\{x := N_{[n]}\}_{[n][p]} \qquad \text{when } n \geq 0$$

- **Labeled substitution**

$$x^n\{y := P\} = x^n \qquad\qquad x^m_{[n]} = x^p$$

$$y^n\{y := P\} = P_{[n]} \qquad\qquad (\lambda x.M)^m = (\lambda x.M)^p$$

$$(\lambda x.M)^n\{y := P\} = (\lambda x.M\{y := P\})^n \qquad\qquad (MN)^m = (MN)^p$$

$$(MN)^n\{y := P\} = (M\{y := P\}\, N\{y := P\})^n \qquad\qquad \text{where } p = \lfloor m, n \rfloor$$

Another labeled calculus

- Examples:

$$((\lambda x.x^{45})^3 y^4)^{12} \longrightarrow y^2$$

$$((\lambda f.(f^9 a^7)^5)^2((\lambda x.x^{45})^3)^{12} \longrightarrow ((\lambda x.x^{45})^1 a^7)^1 \longrightarrow a^0$$

$$((\lambda f.(f^9 a^7)^5)^1((\lambda x.x^{45})^3)^{12} \longrightarrow ((\lambda x.x^{45})^0 a^7)^0$$

$$((\lambda x.(x^9 x^7)^5)^2(\lambda x.(x^9 x^7)^5)^2)^{12} \longrightarrow ((\lambda x.(x^9 x^7)^5)^1(\lambda x.(x^9 x^7)^5)^1)^1$$

$$\longrightarrow ((\lambda x.(x^9 x^7)^5)^0(\lambda x.(x^9 x^7)^5)^1)^0$$

new normal forms

Labeled calculus

- **Theorem** The labeled calculus is **confluent**.

- **Theorem** The labeled calculus is **strongly normalizable** (no infinite labeled reductions).

- **Lemma** For any reduction $\rho : M \xrightarrow{\star} N$, residuals keep degree of redexes

$$\Downarrow$$

- **Theorem 3 [inside-out completeness]:** Any reduction can be overpassed by an inside-out reduction.

Labeled calculus

- **Proof**

  Let $\rho : M \xrightarrow{\star} N$ be any reduction. It can be performed in the labeled calculus by taking large enough exponents of subterms in M. Call U this labeled λ-term. Then $\rho : U \xrightarrow{\star} V$ with M and N being U and V stripped.

  Take any innermost reduction starting from U. It reaches a normal form W since the labeled calculus strongly normalizes.

  This reduction is surely inside-out. If not, a redex inside the one contracted in a previous step has a residual contracted later. Therefore this residual has non-null degree, as redex of which he is residual. Contradicts the fact that ρ was a labeled innermost reduction.

  By Church-Rosser, $V \xrightarrow{\star} W$.

  Let P be W stripped. Then $M \xrightarrow{\star} P$ and $M \xrightarrow{\star} P$ by an inside-out reduction.

- This proof seems magic. But it is an instance of a more general theorem: Generalized finite developments, with the redex family idea (see `[JJL 78]`)

Homeworks

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Exercices

1- Show that if M has no hnf, then M is totally undefined.

2- Show $\Omega M \equiv \Omega$ and $\lambda x.\Omega \equiv \Omega$. Show that $M \longrightarrow_{\omega} N$, then $M \equiv N$.

3- Find M and N such that $MP \equiv NP$ for all P, but $M \not\equiv N$. (Meaning that $\equiv$ is not extensional)

4- Show $M \not\equiv \lambda x.Mx$ when $x \notin \text{var}(M)$. What if $M \equiv \lambda x.M_1$?

5- Let $Y_0 = Y$, $Y_{n+1} = Y_n(\lambda xy.y(xy))$. Show that $Y \equiv Y_n$ for all n. However all Y_n are pairwise non interconvertible.

6 If $M \leq P$ and $N \leq P$ (M and N are prefix compatible), then $\text{BT}(M \sqcap N) = \text{BT}(M) \cap \text{BT}(N)$. (Thus BT is stable in Berry's sense, 1978). What if not compatible ?

Exercices

7- `[Barendregt 1971]`

A closed expression M (i.e. $\mathrm{var}(M) = \emptyset$) is solvable iff:

$$\forall P,\ \exists N_1, N_2, \ldots N_n \text{ such that } M N_1 N_2 \cdots N_n =_\beta P$$

(in short:

$$\forall P,\ \exists \vec{N},\ M\vec{N} =_\beta P \quad)$$

Show that for every closed term M, the following are equivalent:

1. M has a hnf
2. $\exists \vec{N}$, $M\vec{N}$ has a normal form
3. $\exists \vec{N}$, $M\vec{N} =_\beta I$
4. M is solvable

8- `[Barendregt 1974]`

Show that, in the λI-calculus, a term M is solvable iff it has a normal form.

Exercices

9- Let $\mathcal{R}$ be a preorder on $\mathcal{N}$ (reflexive + transitive) compatible with its structure:

$a_1 \mathrel{\mathcal{R}} b_1, \ldots a_n \mathrel{\mathcal{R}} b_n$ implies $x a_1 a_2 \cdots a_n$

$a \mathrel{\mathcal{R}} b$ implies $\lambda x.a \mathrel{\mathcal{R}} \lambda x.b$

Let $M \sqsubseteq_{\mathcal{R}} N$ iff $\forall a \in \mathrm{BT}(M),\ \exists b \in \mathrm{BT}(N),\ a \mathrel{\mathcal{R}} b$

Show that when M is a closed term, one has:

$$\forall \vec{P},\ M\vec{P} \sqsubseteq_{\mathcal{R}} N\vec{P} \ \text{ iff } \ \forall C[\,],\ C[M] \sqsubseteq_{\mathcal{R}} C[N]$$

10- (cont'd 1) Let $M \mathrel{\mathcal{R}} N$ be "if M has a normal form, then N has a normal form"

Give examples of M and N such that $M \sqsubseteq_{\mathcal{R}} N$ but $M \not\sqsubseteq N$.

11- (cont'd 2) Let $M \mathrel{\mathcal{R}} N$ be "if M has a hnf, then N has a hnf"

Give examples of M and N such that $M \sqsubseteq_{\mathcal{R}} N$ but $M \not\sqsubseteq N$.

12- (cont'd 2) Let $M \mathrel{\mathcal{R}} N$ be "if M has a hnf, then N has a similar hnf"

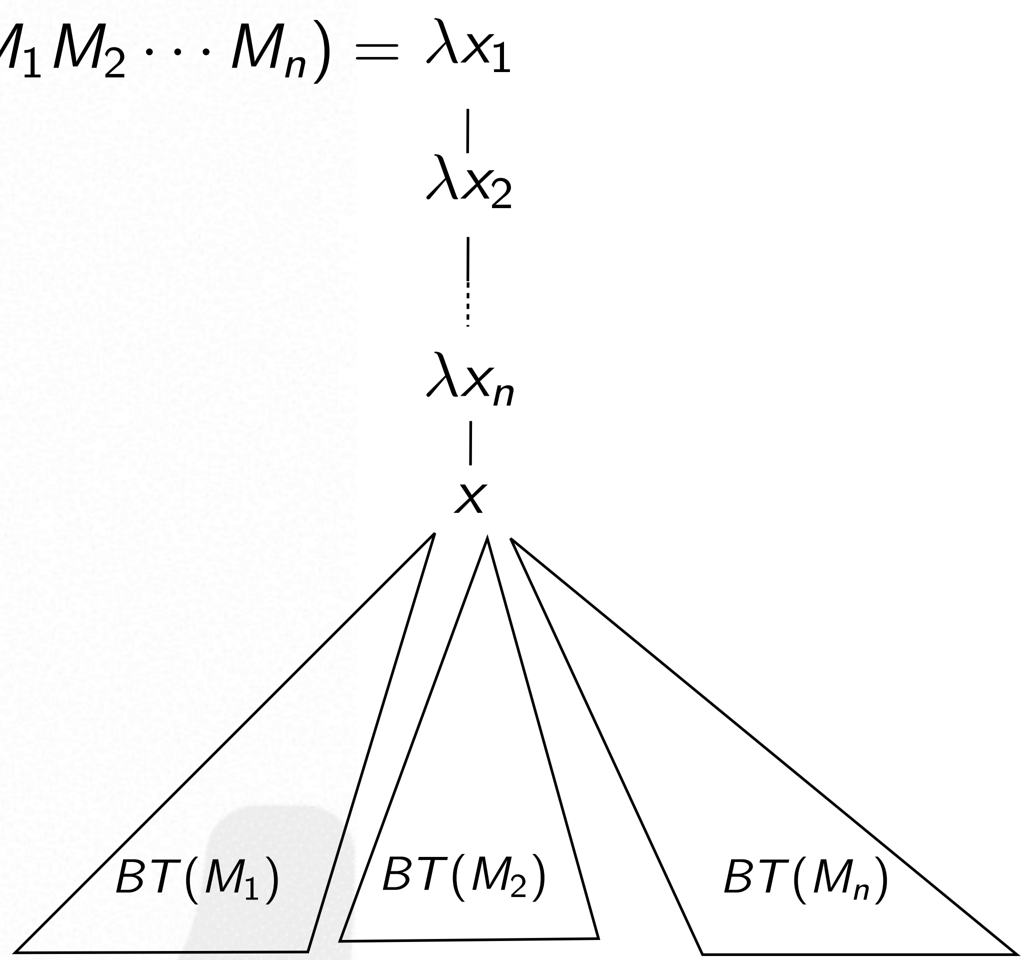
Give examples of M and N such that $M \sqsubseteq_{\mathcal{R}} N$ but $M \not\sqsubseteq N$. (Hint: consider $M = \lambda x.xx$ and $N = \lambda x.x(\lambda y.xy)$) [Compare with `Hyland 1975`])

Exercices

13- Lévy-Longo trees [JJL,1974;GL,1978]

Bohm tree construction can also be done by separating Ω and $\lambda x.\Omega$. Therefore trees will be labeled as follows:

$$LLT(\lambda x_1 x_2 \cdots x_n . x M_1 M_2 \cdots M_n) =$$



Redo all theory with LL-trees. What is LL-tree of YK?