Lambda-Calculus (III-2)

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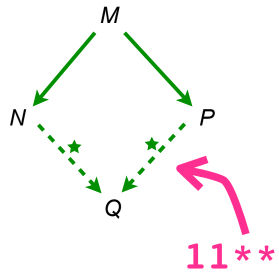
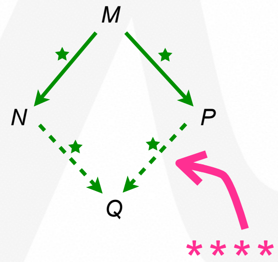
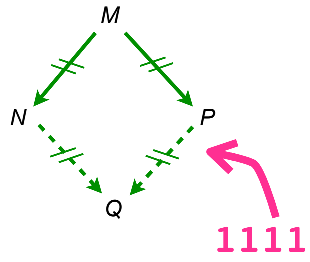
Reminders

- **Local** confluency of β-conversion (lemma `11**`)
- Local confluency ~~$\Rightarrow$~~ **full** confluency
- need for defining **parallel** reduction (lemma `1111`)
- then full confluency (Church-Rosser thm `****`)
- interconvertibility (β-equality) is **consistent**

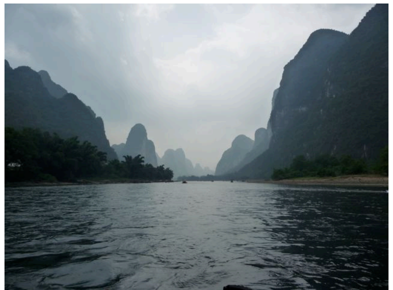






Plan

- Residuals of redexes
- Finite developments theorem
- A labeled calculus ``underlined method''
- Proof of finite developments



Finite developments

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Residuals of redexes

- tracking redexes while contracting others
- examples:

$\Delta(Ia) \longrightarrow Ia(Ia)$ $\qquad \Delta = \lambda x.xx \quad I = \lambda x.x \quad K = \lambda xy.x$

$Ia(\Delta(Ib)) \longrightarrow Ia(Ib(Ib))$

$I(\Delta(Ia)) \longrightarrow I(Ia(Ia))$

$\Delta(Ia) \longrightarrow Ia(Ia))$

$Ia(\Delta(Ib)) \longrightarrow Ia(Ib(Ib))$

$\Delta\Delta \longrightarrow \Delta\Delta$

$(\lambda x.Ia)(Ib) \longrightarrow Ia$

Residuals of redexes

- when ρ is a reduction from M to N, i.e. $\rho : M \xrightarrow{*} N$
  the set of residuals of R by ρ is defined by **transitivity** on the length of ρ and is written R/ρ
- notice that we can have $S \in R/\rho$ and $R \neq S$
  residuals may **not** be syntacticly **equal** (see previous 3rd example)
- residuals **depend upon reduction**. Two reductions between same terms may produce two distinct sets of residuals.
- a redex is residual of a **single** redex (the inverse of the residual relation is a function): $R \in S/\rho$ and $R \in T/\rho$ implies $S = T$

Residuals of redexes

- when R is redex in M and $M \xrightarrow{S} N$
  the set R/S of **residuals** of R in N is defined by inspecting relative positions of R and S in M:

1- R and S disjoint, $M = \cdots R \cdots S \cdots \xrightarrow{S} \cdots R \cdots S' \cdots = N$

2- S in $R = (\lambda x.A)B$

2a- S in A, $M = \cdots (\lambda x. \cdots S \cdots)B \cdots \xrightarrow{S} \cdots (\lambda x. \cdots S' \cdots)B \cdots = N$

2b- S in B, $M = \cdots (\lambda x.A)(\cdots S \cdots) \cdots \xrightarrow{S} \cdots (\lambda x.A)(\cdots S' \cdots) \cdots = N$

3- R in $S = (\lambda y.C)D$

3a- R in C, $M = \cdots (\lambda y. \cdots R \cdots)D \cdots \xrightarrow{S} \cdots \cdots R\{y := D\} \cdots \cdots = N$

3b- R in D, $M = \cdots (\lambda y.C)(\cdots R \cdots) \cdots \xrightarrow{S} \cdots (\cdots R \cdots) \cdots (\cdots R \cdots) \cdots = N$

4- R is S, no residuals of R.

Exercices

- Find redex R and reductions ρ and σ between M and N such that residuals of R by ρ and σ differ. Hint: consider $M = I(Ix)$
- Show that residuals of nested redexes keep nested.
- Show that residuals of disjoint redexes may be nested.
- Show that residuals of a redex may be nested after several reduction steps.

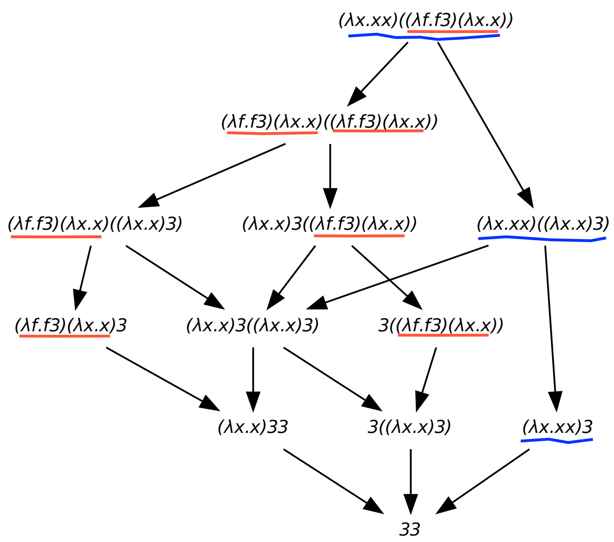
Created redexes

- A redex is **created by reduction** ρ if it is not a residual by ρ of a redex in initial term. Thus R is created by ρ when $\rho : M \xrightarrow{*} N$ and $\nexists S,\ R \in S/\rho$

$(\lambda x.xa)I \longrightarrow Ia$ $\qquad IIa \longrightarrow Ia$

$(\lambda xy.xy)ab \longrightarrow (\lambda y.ay)b$ $\qquad \Delta\Delta \longrightarrow \Delta\Delta$

Residuals of redexes

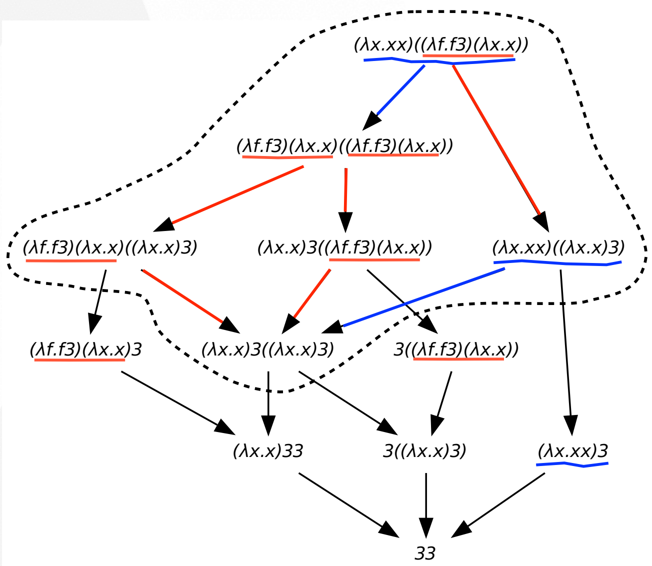


Finite developments

- Let $\mathcal{F}$ be a set of redexes in M. A reduction **relative to** $\mathcal{F}$ only contracts residuals of $\mathcal{F}$.
- When there are no more residuals of $\mathcal{F}$ to contract, we say the relative reduction is a **development of** $\mathcal{F}$.

- **Theorem 3 [finite developments] (Curry)** Let $\mathcal{F}$ be a set of redexes in M. Then:
  - relative reductions **cannot be infinite**; they all end in a development of $\mathcal{F}$
  - all developments end on a **same** term N
  - let R be a redex in M. Then **residuals** of R by finite developments of $\mathcal{F}$ are the same.

Relative reductions



Finite developments

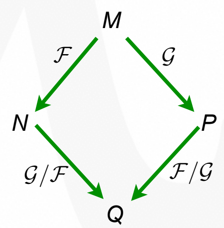
- Therefore we can define (without ambiguity) a new **parallel step** reduction:

$$\rho : M \xrightarrow{\mathcal{F}} N$$

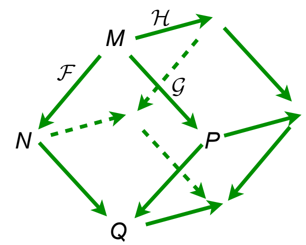
and when R is a redex in M, we can write $R/\mathcal{F}$ for its residuals in N

- **Two corollaries**:

Lemma of **Parallel Moves**



Cube Lemma



Labeled calculus

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Labeled calculus

Take $\mathcal{F} = \{s, u, v\}$ and

$$\left.\begin{array}{lcl} M & = & I^r(\Delta^s(I^t x))(\Delta^u(I^v y)) \\ & \longrightarrow & I^r(I^t x(I^t x))(\Delta^u(I^v y)) \\ & \longrightarrow & I^r(I^t x(I^t x))(\Delta^u y) \\ & \longrightarrow & I^r(I^t x(I^t x))(yy) \end{array}\right\}$$

development of s,u,v

but also

$$\left.\begin{array}{lcl} M & \longrightarrow & I^r(\Delta^s(I^t x))(I^v y(I^v y)) \\ & \longrightarrow & I^r(I^t x(I^t x))(I^v yy) \\ & \longrightarrow & I^r(I^t x(I^t x))(yy) \end{array}\right\}$$ also development of s,u,v

$I = \lambda x.x \qquad \Delta = \lambda x.xx$

Labeled calculus

- Finite developments will be shown with a labeled calculus.
- **Lambda calculus with labeled redexes**

M, N, P	$::=$	$x, y, z, \ldots$	(variables)
	$\mid$	$(\lambda x.M)$	(M as function of x)
	$\mid$	$(M\ N)$	(M applied to N)
	$\mid$	$c, d, \ldots$	(*constants*)
	$\mid$	$(\lambda x.M)^r N$	(*labeled redexes*)

new!

- **$\mathcal{F}$-labeled reduction**

$$(\lambda x.M)^r N \longrightarrow M\{x := N\} \qquad \text{when } r \in \mathcal{F}$$

- **Labeled substitution**

. . . as before

$$((\lambda x.M)^r N)\{y := P\} = ((\lambda x.M)\{y := P\})^r(N\{y := P\})$$

Labeled calculus

- **Theorem** For any $\mathcal{F}$, the labeled calculus is **confluent**.
- **Theorem** For any $\mathcal{F}$, the labeled calculus is **strongly normalizable** (no infinite labeled reductions).
- **Lemma** For any $\mathcal{F}$-reduction $\rho : M \xrightarrow{*} N$, a labeled redex in N has label r if and only if it is **residual** by ρ of a redex with label r in M.

$\Downarrow$

- **Theorem 3 [finite developments] (Curry)**

Labeled calculus proofs

- **Definition [$\mathcal{F}$-labeled parallel reduction]:**

[Var Axiom] $x \mathrel{+\!\!\!\!\!\rightarrow} x$ [Const Axiom] $c \mathrel{+\!\!\!\!\!\rightarrow} c$

[App Rule] $\dfrac{M \mathrel{+\!\!\!\!\!\rightarrow} M' \quad N \mathrel{+\!\!\!\!\!\rightarrow} N'}{MN \mathrel{+\!\!\!\!\!\rightarrow} M'N'}$ [Abs Rule] $\dfrac{M \mathrel{+\!\!\!\!\!\rightarrow} M'}{\lambda x.M \mathrel{+\!\!\!\!\!\rightarrow} \lambda x.M'}$

[//App' Rule] $\dfrac{M \mathrel{+\!\!\!\!\!\rightarrow} M' \quad N \mathrel{+\!\!\!\!\!\rightarrow} N'}{(\lambda x.M)^r N \mathrel{+\!\!\!\!\!\rightarrow} (\lambda x.M')^r N'}$

new!

[//Beta Rule] $\dfrac{M \mathrel{+\!\!\!\!\!\rightarrow} M' \quad N \mathrel{+\!\!\!\!\!\rightarrow} N' \quad r \in \mathcal{F}}{(\lambda x.M)^r N \mathrel{+\!\!\!\!\!\rightarrow} M'\{x := N'\}}$

Labeled calculus proofs

- Proof of confluency is again with Martin-Löf's axiomatic method.
- Proof of residual property is by simple inspection of a reduction step.
- Proof of termination is slightly more complex with following lemmas:

- **Notation** $M \xrightarrow[\text{int}]{*} N$ if M reduces to N without contracting a toplevel redex.

- **Lemma 1** **[Barendregt-like]** $M\{x := N\} \xrightarrow[\text{int}]{*} (\lambda y.P)^r Q$ implies

  $M = (\lambda y.A)^r B$ with $A\{x := N\} \xrightarrow{*} P$, $B\{x := N\} \xrightarrow{*} Q$

  or

  $M = x$ and $N \xrightarrow{*} (\lambda y.P)^r Q$

- **Lemma 2** $M, N \in \mathcal{SN}$ (strongly normalizing) implies $M\{x := N\} \in \mathcal{SN}$

- **Theorem** $M \in \mathcal{SN}$ for all M.

Labeled calculus proofs

- **Substitution lemma:** $M\{x := N\}\{y := P\} \ = \ M\{y := P\}\{x := N\{y := P\}\}$

  when x not free in P

Proof: Induction on $||M||$. Cases 1-4 are as in the unlabeled calculus.

Case 5: $M = (\lambda z.M_1)^r M_2$. This case is easy. Write $A^* = A\{x := N\}\{y := P\}$ and $A^\dagger = A\{y := P\}\{x := N\{y := P\}\}$ for any A.

We have $M^* = ((\lambda z.M_1)^*)^r M_2^* = ((\lambda z.M_1)^\dagger)^r M_2^\dagger$ by induction. Thus again $M^* = M^\dagger$.

QED

Labeled calculus proofs

- **Lemma 1** **[Barendregt-like]** $M\{x := N\} \xrightarrow[\text{int}]{*} (\lambda y.P)^r Q$ implies

  $M = (\lambda y.A)^r B$ with $A\{x := N\} \xrightarrow{*} P$, $B\{x := N\} \xrightarrow{*} Q$

  or

  $M = x$ and $N \xrightarrow{*} (\lambda y.P)^r Q$

Proof Let P^* be $P\{x := N\}$ for any P.

Case 1: $M = x$. Then $M^* = N$ and $N \xrightarrow{*} (\lambda y.P)^r Q$.

Case 2: $M = y$. Then $M^* = y$. Impossible.

Case 2: $M = \lambda y.M_1$. Again impossible.

Case 3: $M = M_1 M_2$ or $M = (\lambda y.M_1)^s M_2$ with $s \neq r$. These cases are also impossible.

Case 4: $M = (\lambda y.M_1)^r M_2$. Then $M_1^* \xrightarrow{*} P$ and $M_2^* \xrightarrow{*} Q$.

QED

Labeled calculus proofs

- **Lemma 2** $M, N \in \mathcal{SN}$ (strongly normalizing) implies $M\{x := N\} \in \mathcal{SN}$

Proof: by induction on $\langle \mathrm{depth}(M), ||M|| \rangle$. Let P^* be $P\{x := N\}$ for any P.

Case 1: $M = x$. Then $M^* = N \in \mathcal{SN}$. If $M = y$. Then $M^* = y \in \mathcal{SN}$.

Case 2: $M = \lambda y.M_1$. Then $M^* = \lambda y.M_1^*$ and by induction $M_1^* \in \mathcal{SN}$.

Case 3: $M = M_1 M_2$ and never $M^* \xrightarrow{*} (\lambda y.A)^r B$. Same argument on M_1 and M_2.

Case 4: $M = M_1 M_2$ and $M^* \xrightarrow{*} (\lambda y.A)^r B$. We can always consider first time when this toplevel redex appears. Hence we have $M^* \xrightarrow[\text{int}]{*} (\lambda y.A)^r B$. By lemma 1, we have two cases:

Case 4.1: $M = (\lambda y.M_3)^r M_2$ with $M_3^* \xrightarrow{*} A$ and $M_2^* \xrightarrow{*} B$. Then $M^* = (\lambda y.M_3^*)^r M_2^*$. As $M_3 \in \mathcal{SN}$ and $M_2 \in \mathcal{SN}$, the internal reductions from M^* terminate by induction. If $r \notin \mathcal{F}$, there are no extra reductions. If $r \in \mathcal{F}$, we can have $M_3^* \xrightarrow{*} A$, $M_2^* \xrightarrow{*} B$ and $(\lambda y.A)^r B \longrightarrow A\{y := B\}$. But $M \longrightarrow M_3\{y := M_2\}$ and $(M_3\{y := M_2\})^* \xrightarrow{*} A\{y := B\}$. As $\mathrm{depth}(A\{y := B\} \leq \mathrm{depth}(M_3\{y := M_2\}) < \mathrm{depth}(M)$, we get $A\{y := B\} \in \mathcal{SN}$ by induction.

Case 4.2: $M = x$. Impossible.

QED

Labeled calculus proofs

- **Theorem** $M \in \mathcal{SN}$ for all M.

Proof: by induction on $||M||$.

Case 1: $M = x$. Obvious.

Case 2: $M = \lambda x.M_1$. Obvious since $M_1 \in \mathcal{SN}$ by induction.

Case 3: $M = M_1 M_2$ and $M_1 \neq (\lambda x.A)^r$. Then all reductions are internal to M_1 and M_2. Therefore $M \in \mathcal{SN}$ by induction on M_1 and M_2.

Case 4: $M = (\lambda x.M_1)^r M_2$ and $r \notin \mathcal{F}$. Same argument on M_1 and M_2.

Case 5: $M = (\lambda x.M_1)^r M_2$ and $r \in \mathcal{F}$. Then M_1 and M_2 in $\mathcal{SN}$ by induction. But we can also have $M \xrightarrow{*} (\lambda x.A)^r B \longrightarrow A\{x := B\}$ with A and B in $\mathcal{SN}$. By Lemma 2, we know that $A\{x := B\} \in \mathcal{SN}$.

QED

Labeled calculus proofs

Case 4.1 (bis): still by induction on $\langle \mathrm{depth}(M), ||M|| \rangle$.

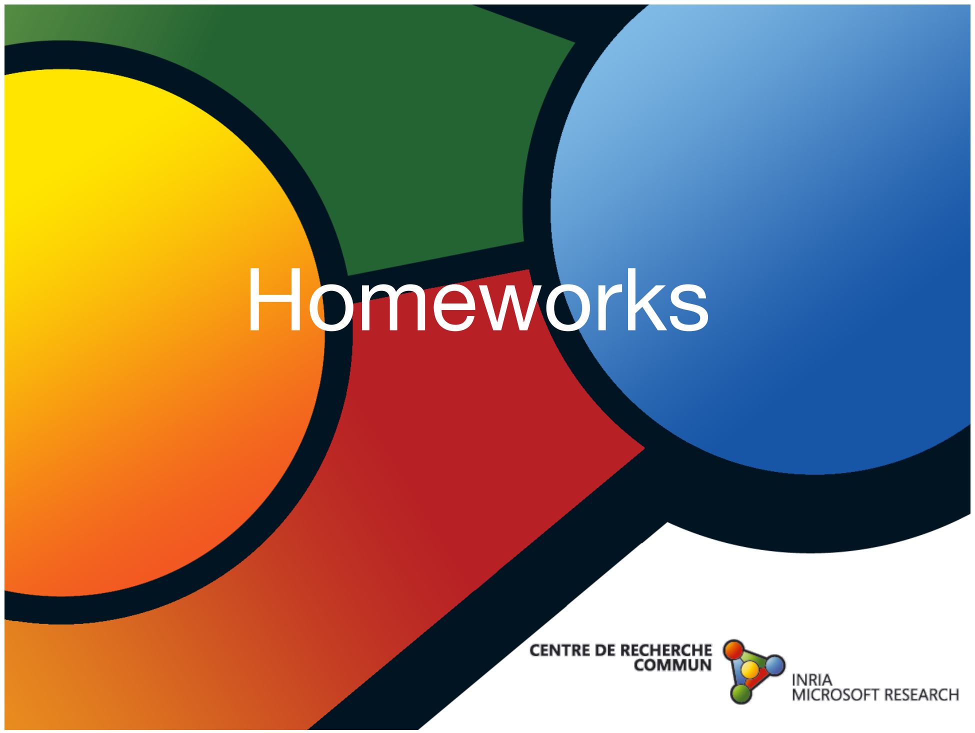
$$\begin{tikzcd}
& M^* = (\lambda y.M_3^*)^r M_2^* \arrow[dl, "*"', "\text{int}"] \arrow[dr] & & M = (\lambda y.M_3)^r M_2 \qquad r \in \mathcal{F} \arrow[d] \\
(\lambda y.A)B \arrow[dr] & & M_3^*\{y := M_2*\} = (M_3\{y := M_2\})^* \arrow[dl, "*"'] & M_3\{y := M_2\} \\
& A\{y := B\} & &
\end{tikzcd}$$

We need substitution lemma and main lemma of Martin-Löf's axiomatic method:

$M\{x := N\}\{y := P\} \;=\; M\{y := P\}\{x := N\{y := P\}\}$ when x not free in P

$M \xrightarrow{/\!\!/} M'$ and $N \xrightarrow{/\!\!/} N'$ implies $M\{x := N\} \xrightarrow{/\!\!/} M'\{x := N'\}$

(in last one, one can replace $\xrightarrow{/\!\!/}$ by $\xrightarrow{*}$)



Exercices

1- Show there is no M such that $M \xrightarrow{*} Kac$ and $M \xrightarrow{*} Kbc$ where $K = \lambda x.\lambda y.x$.

2- Find M such that $M \xrightarrow{*} Kab$ and $M \xrightarrow{*} Kac$.

3- (difficult) Show that $\xleftarrow{*}$ is not confluent.

4- Show there is no M whose reduction graph is exactly following:

$$\begin{tikzcd}
& M \arrow[dl] \arrow[d] \arrow[dr] & \\
M_1 \arrow[dr] & M_2 \arrow[d] & M_3 \arrow[dl] \\
& N &
\end{tikzcd}$$

5- Show there is no M such that $M \xrightarrow{*} \lambda x.N$ and $M \xrightarrow{*} yM_1M_2 \cdots M_n$.

6- Show there is no M such that $M \xrightarrow{*} xN_1N_2 \cdots N_n$ and $M \xrightarrow{*} yP_1P_2 \cdots P_n$ $(x \neq y)$.

7- Show that $\xleftarrow{*}_\eta$ and $(\longrightarrow \cup \longleftarrow_\eta)^*$ are confluent.

Exercices

8- Equivalence by permutations.