Lambda-Calculus (III-1)

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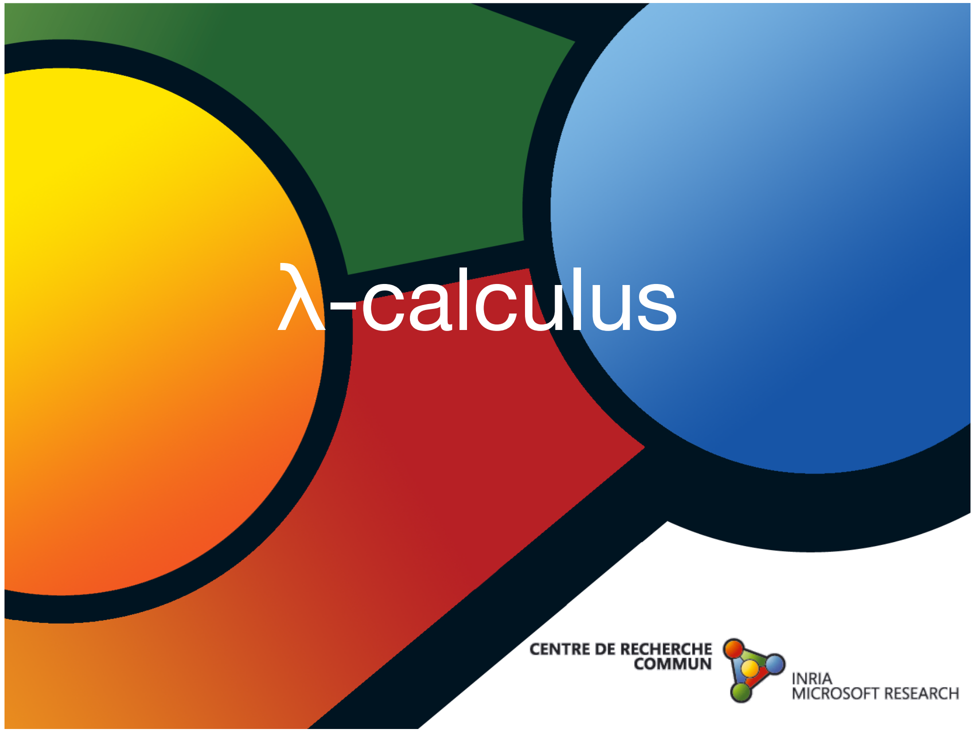


Plan

- language
- abbreviations
- local confluency
- Church Rosser theorem
- Redexes and residuals

The lambda-calculus

- Lambda terms

M, N, P	::=	*x, y, z, ...*	(variables)
	|	(λ*x.M*)	(*M* as function of *x*)
	|	(*M N*)	(*M* applied to *N*)
	|	*c, d, ...*	(*constants*)

- Calculations “reductions”

$$((\lambda x.M)N) \longrightarrow M\{x := N\}$$

Abbreviations

$MM_1M_2\cdots M_n$ for $(\cdots((MM_1)M_2)\cdots M_n)$

$(\lambda x_1x_2\cdots x_n\,.\,M)$ for $(\lambda x_1.(\lambda x_2.\cdots(\lambda x_n\,.\,M)\cdots))$

external parentheses and parentheses after a dot may be forgotten

Exercice 1

Write following terms in long notation:

$\lambda x.x,\ \lambda x.\lambda y.x,\ \lambda xy.x,\ \lambda xyz.y,\ \lambda xyz.zxy, \lambda xyz.z(xy),$
$(\lambda x.\lambda y.x)MN,\ (\lambda xy.x)MN,\ (\lambda xy.y)MN,\ (\lambda xy.y)(MN)$

Substitution

$x\{y := P\} = x$ $\qquad c\{y := P\} = c$
$y\{y := P\} = P$
$(MN)\{y := P\} = M\{y := P\}\ N\{y := P\}$
$(\lambda y.M)\{y := P\} = \lambda y.M$
$(\lambda x.M)\{y := P\} = \lambda x'.M\{x := x'\}\{y := P\}$
where $x' = x$ if y not free in M or x not free in P,
otherwise x' is the first variable not free in M and P.
(we suppose that the set of variables is infinite and enumerable)

Free variables

$\mathrm{var}(x) = \{x\}$ $\qquad \mathrm{var}(c) = \emptyset$
$\mathrm{var}(MN) = \mathrm{var}(M) \cup \mathrm{var}(N)$
$\mathrm{var}(\lambda x.M) = \mathrm{var}(M) - \{x\}$

Examples

$(\lambda x.x)N \longrightarrow N$

$(\lambda f.f\ N)(\lambda x.x) \longrightarrow (\lambda x.x)N \longrightarrow N$

$(\lambda x.xx)(\lambda x.xN) \longrightarrow (\lambda x.xN)(\lambda x.xN) \longrightarrow (\lambda x.xN)N \longrightarrow NN$

$(\lambda x.xx)(\lambda x.xx) \longrightarrow (\lambda x.xx)(\lambda x.xx) \longrightarrow \cdots$

$Y_f = (\lambda x.f(xx))(\lambda x.f(xx)) \longrightarrow f((\lambda x.f(xx))(\lambda x.f(xx))) = f(Y_f)$

$f(Y_f) \longrightarrow f(f(Y_f)) \longrightarrow \cdots \longrightarrow f^n(Y_f) \longrightarrow \cdots$

Conversion rules

$$\begin{array}{rcll} \lambda x.M & \longrightarrow_\alpha & \lambda x'.M\{x := x'\} & (x' \notin \mathrm{var}(M)) \\ (\lambda x.M)N & \longrightarrow_\beta & M\{x := N\} & \\ \lambda x.Mx & \longrightarrow_\eta & M & (x \notin \mathrm{var}(M)) \end{array}$$

- left-hand-side of conversion rule is a **redex** (reductible expression)
- α-redex, β-redex, η-redex, ...
- we forget indices when clear from context, often β

Reduction step

- let R be a redex in M. Then one can contrat redex R in M and get N:

$$M \xrightarrow{R} N$$

Reductions

$$M \xrightarrow{*} N \quad \text{when} \quad M = M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots M_n = N \quad (n \geq 0)$$

- same with explicit contracted redexes

$$M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

- and with named reductions

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

- we speak of redex occurences when specifying reduction steps, but it is convenient to confuse redexes and redex occurences when clear from context

Exercice 3

- Show that $M \longrightarrow N$ implies $\mathrm{var}(N) \subset \mathrm{var}(M)$.
- Find terms M such that:

$$M \longrightarrow M$$
$$M = M_0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow \cdots M_n = M \quad (M_i \text{ all distinct})$$
$$M =_\beta x\, M$$
$$M =_\beta \lambda x.M$$
$$M =_\beta MM$$
$$M =_\beta MN_1N_2 \cdots N_n \quad \text{for all } N_1, N_2, \ldots N_n$$

- Find term Y such that, for any M:

$$YM =_\beta M(YM)$$

- Find Y' such that, for any M:

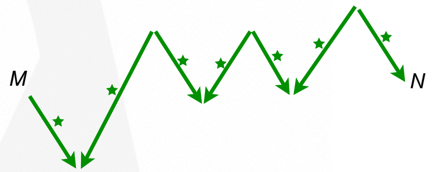
$$Y'M \xrightarrow{*} M(Y'M)$$

- (difficult) Show there is only one redex R such that $R \longrightarrow R$

Lambda theories

$M =_\beta N$ when M and N are related by a zigzag of reductions

M and N are said **interconvertible**



- Also $M =_\alpha N$, $M =_\eta N$, $M =_{\beta,\eta} N, \ldots$
- Interconvertibility is symmetric, reflexive, transivite closure of reduction relation
- or with notations of mathematical logic:

$$\alpha \vdash M = N, \quad \beta \vdash M = N, \quad \eta \vdash M = N, \quad \beta + \eta \vdash M = N, \ldots$$

- the syntactic equality $M = N$ will often stand for $M =_\alpha N$.

Normal forms

- An expression M without redexes **is in** normal form

$$M \not\longrightarrow$$

- If M reduces to a normal form, then M **has a** normal form

$$M \xrightarrow{*} N, \quad N \text{ in normal form}$$

Exercice 4

- which of following terms are in β-normal form ?

in $\beta\eta$-normal form ?

$$\lambda x.x \qquad \lambda x.x(\lambda xy.x)(\lambda x.x)$$
$$\lambda xy.x \qquad \lambda xy.x(\lambda xy.x)(\lambda x.yx)$$
$$\lambda xy.xy \qquad \lambda xy.x((\lambda x.xx)(\lambda x.xx))y$$
$$\lambda xy.x((\lambda x.y(xx))(\lambda x.y(xx)))$$

Exercice 5

- Show that if M is in normal form and $M \xrightarrow{*} N$, then $M = N$
- Show that:

1- $\lambda x.M \xrightarrow{*} N$ implies $N = \lambda x.N'$ and $M \xrightarrow{*} N'$

2- $MN \xrightarrow{*} P$ implies $M \xrightarrow{*} M'$, $N \xrightarrow{*} N'$ and $P = M'N'$
or $M \xrightarrow{*} \lambda x.M'$, $N \xrightarrow{*} N'$ and $M'\{x := N'\} \xrightarrow{*} P$

3- $xM_1M_2 \cdots M_n \xrightarrow{*} N$ implies $M_1 \xrightarrow{*} N_1$, $M_2 \xrightarrow{*} N_2$, ... $M_n \xrightarrow{*} N_n$
and $xN_1N_2 \cdots N_n = N$

4- $M\{x := N\} \xrightarrow{*} \lambda y.P$ implies $M \xrightarrow{*} \lambda y.M'$ and $M'\{x := N\} \xrightarrow{*} P$
or $M \xrightarrow{*} xM_1M_2 \cdots M_n$ and $NM_1\{x := N\} \cdots M_n\{x := N\} \xrightarrow{*} \lambda y.P$

Confluency

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Reduction (axiomatic def.)

We can define reduction $\longrightarrow$ axiomatically by following axioms and rules:

- **Definition [beta reduction]:**

[App1 Rule] $\dfrac{M \longrightarrow M'}{MN \longrightarrow M'N}$ [App2 Rule] $\dfrac{N \longrightarrow N'}{MN \longrightarrow MN'}$

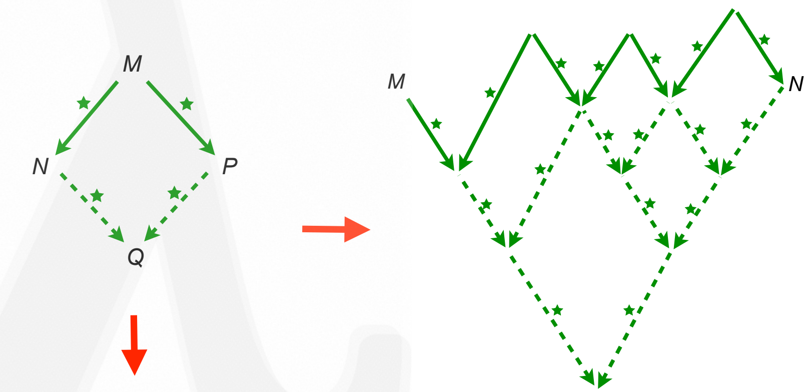
[Abs Rule] $\dfrac{M \longrightarrow M'}{\lambda x.M \longrightarrow \lambda x.M'}$ [//Beta Axiom] $(\lambda x.M)N \longrightarrow M\{x := N\}$

Exercice 6

Give axiomatic definition for $\xrightarrow{*}$, $\longrightarrow_{\eta}$, $\xrightarrow{*}_{\beta,\eta}$.

Confluency

Question: If $M \xrightarrow{*} N$ and $M \xrightarrow{*} P$, then $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$ for some Q ?

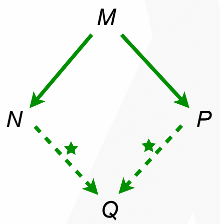


Corollary: [unicity of normal forms]

If $M \xrightarrow{*} N$ in normal form and $M \xrightarrow{*} N'$ in normal form, then $N = N'$.

Local confluency

- **Theorem 1 [`lemma 11**`]:** If $M \longrightarrow N$ and $M \longrightarrow P$, then $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$ for some Q

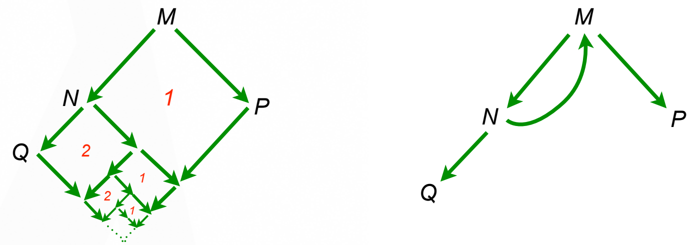


- **Lemma 1:** $M \longrightarrow N$ implies $P\{x := M\} \xrightarrow{*} P\{x := N\}$
- **Lemma 2:** $M \longrightarrow N$ implies $M\{x := P\} \longrightarrow N\{x := P\}$

- **Substitution lemma:** $M\{x := N\}\{y := P\} \;=\; M\{y := P\}\{x := N\{y := P\}\}$ when x not free in P

Confluency

- **Fact:** local confluency does not imply confluency



Local confluency

- **Substitution lemma:** $M\{x := N\}\{y := P\} \;=\; M\{y := P\}\{x := N\{y := P\}\}$ when x not free in P

Proof:

Write $A^* = A\{x := N\}\{y := P\}$ and $A^\dagger = A\{y := P\}\{x := N\{y := P\}\}$ for any A.

Case 1: $M = x$. Then $M^* = N\{y := P\} = M^\dagger$.

Case 2: $M = y$. Then $M^* = P = P\{x := A\}$ for any A since $x \notin \mathrm{var}(P)$. Therefore $M^* = P\{x := N\{y := P\}\} = M^\dagger$.

Case 3: $M = M_1M_2$. This case is easy by induction.

Case 4: $M = \lambda z.M_1$. We assume (by α-conversion) that z is a fresh variable neither in N, nor in P. Then induction is easy, since $M^* = \lambda z.M_1^*$ and $M^\dagger = \lambda z.M_1^\dagger$. This case is then similar to the previous one.

Confluency

We define $\mathrel{/\!\!/\!\!\!\rightarrow}$ such that $\longrightarrow \;\subset\; \mathrel{/\!\!/\!\!\!\rightarrow} \;\subset\; \xrightarrow{*}$

- **Definition [parallel reduction]:**

[Var Axiom] $x \mathrel{/\!\!/\!\!\!\rightarrow} x$

[Const Axiom] $c \mathrel{/\!\!/\!\!\!\rightarrow} c$

[App Rule] $\dfrac{M \mathrel{/\!\!/\!\!\!\rightarrow} M' \quad N \mathrel{/\!\!/\!\!\!\rightarrow} N'}{MN \mathrel{/\!\!/\!\!\!\rightarrow} M'N'}$

[Abs Rule] $\dfrac{M \mathrel{/\!\!/\!\!\!\rightarrow} M'}{\lambda x.M \mathrel{/\!\!/\!\!\!\rightarrow} \lambda x.M'}$

[//Beta Rule] $\dfrac{M \mathrel{/\!\!/\!\!\!\rightarrow} M' \quad N \mathrel{/\!\!/\!\!\!\rightarrow} N'}{(\lambda x.M)N \mathrel{/\!\!/\!\!\!\rightarrow} M'\{x := N'\}}$

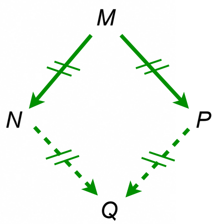
- Example:

$$\dfrac{\dfrac{x \mathrel{/\!\!/\!\!\!\rightarrow} x \quad z \mathrel{/\!\!/\!\!\!\rightarrow} z}{Iz \mathrel{/\!\!/\!\!\!\rightarrow} z} \qquad \dfrac{x \mathrel{/\!\!/\!\!\!\rightarrow} x \quad z \mathrel{/\!\!/\!\!\!\rightarrow} z}{Iz \mathrel{/\!\!/\!\!\!\rightarrow} z}}{Iz(Iz) \mathrel{/\!\!/\!\!\!\rightarrow} zz}$$

$I = \lambda x.x$

Confluency

- Goal is to prove **strongly local confluency:**



- Example: $(\lambda x.xx)(Iz) \xrightarrow{/\!/} (\lambda x.xx)z \xrightarrow{/\!/} zz$
  $(\lambda x.xx)(Iz) \xrightarrow{/\!/} Iz(Iz) \xrightarrow{/\!/} zz$

Confluency

- **Lemma 4:** $M \xrightarrow{/\!/} N$ and $P \xrightarrow{/\!/} Q$ implies $M\{x := P\} \xrightarrow{/\!/} N\{x := Q\}$
- **Lemma 5:** If $M \xrightarrow{/\!/} N$ and $M \xrightarrow{/\!/} P$, then $N \xrightarrow{/\!/} Q$ and $N \xrightarrow{/\!/} Q$ for some Q.

  Proofs L6/L7: structural induction + substitution lemma.

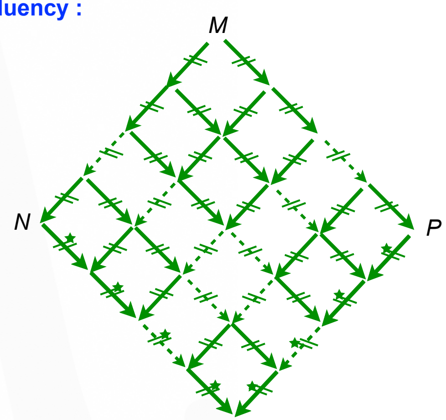
- **Lemma 6:** If $M \longrightarrow N$, then $M \xrightarrow{/\!/} N$.
- **Lemma 7:** If $M \xrightarrow{/\!/} N$, then $M \xrightarrow{*} N$.

  Proofs L6/L7: obvious.

- **Theorem 2 [Church-Rosser]:**
  If $M \xrightarrow{*} N$ and $M \xrightarrow{*} P$, then $N \xrightarrow{*} Q$ and $P \xrightarrow{*} Q$ for some Q.

Confluency

- **Proof of confluency :**



Confluency

- previous axiomatic method is due to **Tait** and **Martin-Löf**
- Tait--Martin-Löf's method models inside-out parallel reductions
- there are other proofs with explicit redexes

- Curry's finite developments

Residuals of redexes

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Residuals of redexes

- when R is redex in M and $M \xrightarrow{S} N$

  the set R/S of **residuals** of R in N is defined by inspecting relative positions of R and S in M:

1- R and S disjoint, $M = \cdots R \cdots S \cdots \xrightarrow{S} \cdots R \cdots S' \cdots = N$

2- S in $R = (\lambda x.A)B$

- **2a-** S in A, $M = \cdots (\lambda x. \cdots S \cdots)B \cdots \xrightarrow{S} \cdots (\lambda x. \cdots S' \cdots)B \cdots = N$
- **2b-** S in B, $M = \cdots (\lambda x.A)(\cdots S \cdots) \cdots \xrightarrow{S} \cdots (\lambda x.A)(\cdots S' \cdots) \cdots = N$

3- R in $S = (\lambda y.C)D$

- **3a-** R in C, $M = \cdots (\lambda y. \cdots R \cdots)D \cdots \xrightarrow{S} \cdots\cdots R\{y := D\} \cdots\cdots = N$
- **3b-** R in D, $M = \cdots (\lambda y.C)(\cdots R \cdots) \cdots \xrightarrow{S} \cdots (\cdots R \cdots) \cdots (\cdots R \cdots) \cdots = N$

4- R is S, no residuals of R.

Residuals of redexes

- tracking redexes while contracting others
- examples:

$\Delta(Ia) \longrightarrow Ia(Ia)$ $\qquad \Delta = \lambda x.xx \quad I = \lambda x.x \quad K = \lambda xy.x$

$Ia(\Delta(Ib)) \longrightarrow Ia(Ib(Ib))$

$I(\Delta(Ia)) \longrightarrow I(Ia(Ia))$

$\Delta(Ia) \longrightarrow Ia(Ia))$

$Ia(\Delta(Ib)) \longrightarrow Ia(Ib(Ib))$

$\Delta\Delta \longrightarrow \Delta\Delta$

$(\lambda x.Ia)(Ib) \longrightarrow Ia$

Residuals of redexes

- when ρ is a reduction from M to N, i.e. $\rho : M \xrightarrow{*} N$

  the set of residuals of R by ρ is defined by **transitivity** on the length of ρ and is written R/ρ

- notice that we can have $S \in R/\rho$ and $R \neq S$

  residuals may **not** be syntacticly **equal** (see previous 3rd example)

- residuals **depend on reductions**. Two reductions between same terms may produce two distinct sets of residuals.

- a redex is residual of a **single** redex (the inverse of the residual relation is a function): $R \in S/\rho$ and $R \in T/\rho$ implies $S = T$

Exercice 7

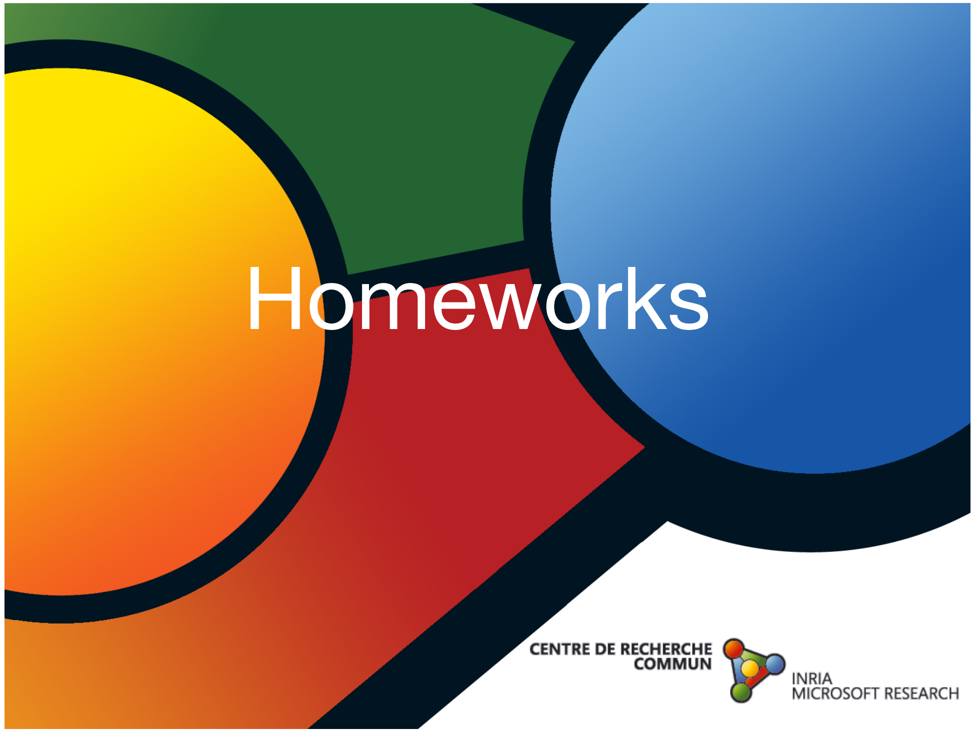
- Find redex R and reductions ρ and σ between M and N such that residuals of R by ρ and σ differ. Hint: consider $M = I(Ix)$
- Show that residuals of nested redexes keep nested.
- Show that residuals of disjoint redexes may be nested.
- Show that residuals of a redex may be nested after several reduction steps.

Created redexes

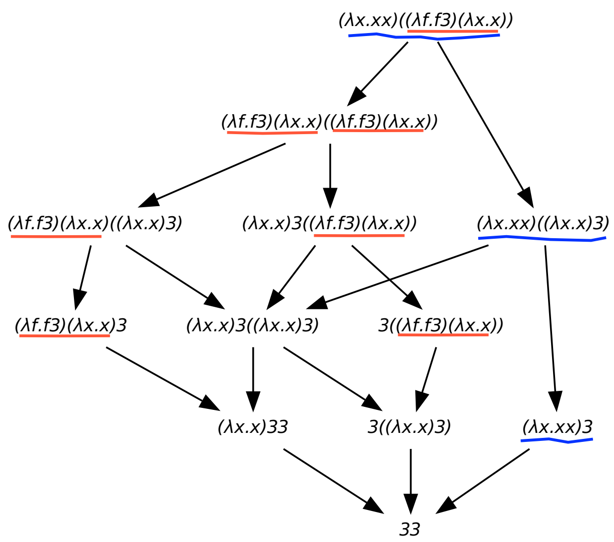
- A redex is **created by reduction** ρ if it is not a residual by ρ of a redex in initial term. Thus R is created by ρ when $\rho : M \xrightarrow{*} N$ and $\nexists S,\ R \in S/\rho$

$$(\lambda x.xa)I \longrightarrow Ia \qquad\qquad IIa \longrightarrow Ia$$

$$(\lambda xy.xy)ab \longrightarrow (\lambda y.ay)b \qquad\qquad \Delta\Delta \longrightarrow \Delta\Delta$$



Residuals of redexes



Exercices 8

- Show that:

  1- $M \longrightarrow_\eta N \longrightarrow P$ implies $M \longrightarrow Q \xrightarrow{*}_\eta P$ for some Q

  2- $M \xrightarrow{*}_\eta N \xrightarrow{*} P$ implies $M \xrightarrow{*} Q \xrightarrow{*}_\eta P$ for some Q

  3- $M \xrightarrow{*}_{\beta,\eta} N$ implies $M \xrightarrow{*} P \xrightarrow{*}_\eta N$ for some P

  4- $M \longrightarrow N$ and $M \longrightarrow_\eta P$ implies $N \xrightarrow{*}_\eta Q$ and $P \xrightarrow{1} Q$ for some Q

  5- $M \xrightarrow{*}_\eta N$ and $M \xrightarrow{*}_\eta P$ implies $N \xrightarrow{*}_\eta Q$ and $P \xrightarrow{*}_\eta Q$ for some Q

  6- $M \xrightarrow{*}_{\beta,\eta} N$ and $M \xrightarrow{*}_{\beta,\eta} P$ implies $N \xrightarrow{*}_{\beta,\eta} Q$ and $P \xrightarrow{*}_{\beta,\eta} Q$ for some Q

  Therefore $\xrightarrow{*}_{\beta,\eta}$ is confluent.

- Show same property for β-reduction and η-expansion $(\longrightarrow \cup \longleftarrow_\eta)^*$

Exercices

7- Show there is no M such that $M \xrightarrow{*} Kac$ and $M \xrightarrow{*} Kbc$ where $K = \lambda x.\lambda y.x$.

8- Find M such that $M \xrightarrow{*} Kab$ and $M \xrightarrow{*} Kac$.

9- (difficult) Show that $\xleftarrow{*}$ is not confluent.