Eklof's chapter discusses the ultraproduct operation, its relation with first-order logic, and its positive applications to algebra. Macintyre's chapter discusses both positive and negative applications to algebra of Abraham Robinson's notion of *model complete theory* and related concepts of "algebraically closed"

Morley's chapter on homogenous sets discusses so-called Ehrenfeucht–Mostowski models. This construction has proven extremely useful in model theory and in applications to set theory. It has had some applications to other parts of mathematics, but should have more once it becomes better known.

To date the principal application of model theory outside algebra and set theory comes from Robinson's "nonstandard analysis". Stroyan's chapter discusses elementary aspects of the subject and gives a more advanced case study of the hidden role infinitesimals play in differential geometry.

The last three chapters in Part A go beyond ordinary first-order logic. Some extensions of first-order logic are mentioned in the last section of Barwise's chapter and discussed in more detail in the last section of Keisler's chapter. Of all the known extensions, the logic $L_{\omega_1\omega}$ has the smoothest model theory. This logic, and its admissible fragments, are discussed in Makkai's chapter.

The final chapter, by Kock and Reyes, is quite different in character. It gives the category theoretical point of view of some topics from model theory and other parts of logic.

It was planned to have a chapter on stability theory and one on abstract model theory. This proved impossible so stability theory is now surveyed in Section 8 of Keisler's chapter. Abstract model theory is discussed at the end of Barwise's chapter and is touched on in Keisler's chapter. Among the other chapters of the Handbook which are particularly relevant to model theory are Rabin's chapter on decidable and undecidable theories, and Aczel's chapter on inductive definitions, both in Part C of the book.

A.1

An Introduction to First-Order Logic

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Contents

* We would like to express our gratitude to the Alfred P. Sloan Foundation and the National Science Foundation (MPS 74-06355A1) for support.

HANDBOOK OF MATHEMATICAL LOGIC
Edited by J. Barwise

1. Foreword

This introductory chapter is written for the (less than ideal) mathematician who knows next to nothing about mathematical logic, and is entirely expository in nature. This might be someone who tried to read a later chapter but got bogged down simply because he did not understand the basic notions. For most readers a quick reading of Section 2 and the introductions to Sections 4 and 5 should suffice.

Modern mathematics might be described as the science of abstract objects, be they real numbers, functions, surfaces, algebraic structures or whatever. Mathematical logic adds a new dimension to this science by paying attention to the language used in mathematics, to the ways abstract objects are defined, and to the laws of logic which govern us as we reason about these objects. The logician undertakes this study with the hope of understanding the phenomena of mathematical experience and eventually contributing to mathematics, both in terms of important results that arise out of the subject itself (Gödel's Second Incompleteness Theorem is the most famous example) and in terms of applications to other branches of mathematics. The chapters of this Handbook are intended to illustrate both of these aspects of mathematical logic.

Modern mathematical logic has its origins in the dream of Leibniz of a universal symbolic calculus which could encompass all mental activity of a logically rigourous nature, in particular, all of mathematics. This vision was too grandiose for Leibniz to realize. His writings on the subject were largely forgotten and had little influence on the actual course of events. It took Boole, Frege, Peano, Russell and Whitehead, Hilbert, Skolem, Gödel, Tarski and their followers, armed with more powerful abstract methods, and motivated (at least in the case of Russell and Hilbert) by apparent problems in the foundations of mathematics, to realize a significant part of Leibniz' dream.

2. How to tell if you are in the realm of first-order logic

Our goal in this section is quite modest: to give the reader, by means of examples, a feeling for what can and what cannot be expressed in first-order logic. Most of our examples are taken from the wealth of notions in modern algebra with which most mathematicians have at least a nodding acquaintance.

The basic building blocks of first-order logic consist of the logical

connectives: $\wedge$ (and), $\vee$ (or), $\neg$ (not), $\rightarrow$ (implies), the equality symbol $=$, quantifiers $\forall$ (for all), $\exists$ (there exists) plus an infinite sequence of variables $x, y, z, x_1, y_1, \ldots$ and some parentheses), (to help the formulas stay readable.

In addition to these logical symbols, a set L of primitive non-logical symbols is given by the topic under discussion. For example, if we are working with abelian groups then the set L has a function symbol $+$ for group addition and a constant symbol 0 for the zero element. If we are working with orderings, then L has a relation symbol $<$. For the study of set theory, L has a relation symbol $\in$. We will postpone the rather tedious formal definition of formula of first-order logic until the next section. Here we stress only that formulas are certain *finite* strings of symbols.

The "first" in the phrase "first-order logic" is there to distinguish this form of logic from stronger logics (like second-order or weak second-order logic) where certain extralogical notions (like set or natural number) are taken as given in advance. In particular, in first-order logic the quantifiers $\forall$ and $\exists$ always range over elements of the domain M of discourse. By contrast, second-order logic allows one to quantify over subsets of M and functions F mapping, say, $M \times M$ into M. (Third-order logic goes on to sets of functions, etc.) Weak second-order logic allows quantification over finite subsets of M and over natural numbers. There are good reasons for considering first-order logic to be the basic language of mathematics; these will be discussed in Section 5. We assume here that the reader has his own motivation for wanting to find out what first-order logic is.

Group theory

Our first few examples come from group theory. Consider the following notions:

(a) group,
(b) abelian group,
(c) abelian group with every element of order $\leq n$,
(d) divisible group,
(e) torsion-free group,
(f) torsion group.

The notions (a)–(c) are easily axiomatized by a few first-order axioms. Notions (d) and (e) take an infinite list of axioms. The last notion (f) is not first-order. Let's see why.

A group G is a triple $G = \langle G, +, 0\rangle$ (where G is a nonempty set, $0 \in G$ and $+$ is a function mapping $G \times G$ into G) which satisfies the following first-order axioms, or sentences:

$$\forall x\, \forall y\, \forall z\, [x + (y + z) = (x + y) + z], \tag{1}$$

$$\forall x\, [x + 0 = x], \tag{2}$$

$$\forall x\, \exists y\, [x + y = 0]. \tag{3}$$

The logician might say that G *is a model of* (1), (2), (3) and write $G \models$ (1), (2), (3), instead of saying that G satisfies (1), (2), (3).

An *abelian group* is a group G satisfying the axiom

$$\forall x\, \forall y\, [x + y = y + x]. \tag{4}$$

The choice of the symbol "+" in (1)–(4) is dictated by convention only; it has no real significance.

To express the next notion we abbreviate the formal term $(x + x)$ by $2x$, the term $((x + x) + x)$ by $3x$ and, by induction, we abbreviate the term $(nx + x)$ by $(n + 1)x$. An abelian group G has every element of order $\leq n$ if G is a model of

$$\forall x\, [x = 0 \vee 2x = 0 \vee \cdots \vee nx = 0]. \tag{5}$$

This is a simple first-order sentence.

An abelian group G is *divisible* if

$$\forall n \geq 1\, \forall x\, \exists y\, [ny = x]. \tag{6}$$

This would count as a sentence of weak second-order logic but it is *not* a first-order axiom because the leading quantifier ranges over the set of positive natural numbers, rather than over the domain of discourse G. We can, however, replace this expression by the following infinite list of axioms:

$$\forall x\, \exists y\, [2y = x], \tag*{$(6)_2$}$$

$$\forall x\, \exists y\, [3y = x], \tag*{$(6)_3$}$$

$$\vdots$$

$$\forall x\, \exists y\, [ny = x], \tag*{$(6)_n$}$$

(We left off $(6)_1$ since it is the trivial sentence $\forall x\, \exists y\, [x = y]$.) For most purposes such an effectively presented infinite list of axioms is practically as good as a finite list. Still, it is worth proving for our own satisfaction that it is not just lack of imagination which forces us to use an infinite list to express the notion.

2.1. PROPOSITION. *Any finite set of first-order sentences true in all divisible abelian groups is true in some nondivisible abelian group.*

In other words, the notion of divisible abelian group is not *finitely* axiomatizable in first-order logic. We delay the proof of this result for a few paragraphs.

We discover essentially the same phenomenon when we attempt to axiomatize the concept of *torsion-free* abelian group:

$$\forall n \geq 1\, \forall x\, [x \neq 0 \rightarrow nx \neq 0]. \tag{7}$$

This sentence of weak second-order logic turns into an infinite list of first-order axioms:

$$\forall x\, [x \neq 0 \rightarrow nx \neq 0]. \tag*{$(7)_n$}$$

We have the corresponding negative result.

2.2. PROPOSITION. *The notion of torsion-free abelian group is not finitely axiomatizable in first-order logic.*

An abelian group G is *torsion* if it satisfies

$$\forall x\, \exists n \geq 1\, [nx = 0]. \tag{8}$$

This is a sentence of weak second-order logic but it is not first-order because it has the quantifier $\exists n$ over natural numbers. We could try to imitate (5) but look what happens:

$$\forall x\, [x = 0 \vee 2x = 0 \vee \cdots \vee nx = 0 \vee \cdots]. \tag*{$(8)'$}$$

This sort of expression is analogous to an infinite formal power series and the study of such idealized "infinitary formulas" has turned out to be quite profitable (see 5.3, and Chapters A.2 and A.7) but it is not part of ordinary first-order logic. To clinch matters we will prove the following result.

2.3. PROPOSITION. *The set of first-order sentences true in all torsion abelian groups is true in some abelian group H which is not torsion.*

In fact, what we will show is that if G is an abelian group with no finite bound on the order of its elements, then there is a group H which is not torsion but such that $G \equiv H$, which means that every first-order sentence true in G is also true in H, and vice versa. Therefore the class of torsion groups cannot be characterized even by a set of first-order axioms — finite or infinite.

Nonaxiomatizability results

There are two standard tools for proving nonaxiomatizability results. They are corollaries of the Completeness Theorem and will be proved in Section 4. We will use these tools to prove all the results of this section.

2.4. COMPACTNESS THEOREM (Gödel–Malcev). *Let T be any set of first-order axioms. If for every finite subset T_0 of T there is a model of all the axioms in T_0, then there is a single model of all the axioms in T.*

An alternate form of the Compactness Theorem is sometimes more convenient. Let us write $T \models \psi$ to indicate that ψ is a *logical consequence of* T in the sense that ψ is true in all models which make all the axioms of T true. Then the Compactness Theorem is equivalent to the statement: *If $T \cup \{\psi\}$ is a set of first order sentences and $T \models \psi$, then there is a finite $T_0 \subseteq T$ such that $T_0 \models \psi$.* To see that this follows from 2.4, apply to 2.4 to $T \cup \{\neg\psi\}$, where $\neg\psi$ asserts that ψ is false. To prove 2.4 from this version, let ψ be some absurd ψ like $\exists x\,(x \neq x)$. The Compactness Theorem *fails* for second-order logic or even weak second-order logic, as the proof of 2.1 will show.

The other property of first-order logic sometimes used to prove nonaxiomatizability results is the following Löwenheim–Skolem Theorem. This important principle also holds for weak second-order logic but not for second-order logic.

2.5. LÖWENHEIM–SKOLEM THEOREM. *Let κ be an infinite cardinal and let T be a set of at most κ first-order axioms. If there is a model making all the axioms in T true, then there is such a model whose set of elements has cardinality $\leq \kappa$.*

Remark. As long as the set L of nonlogical symbols is finite, or even countable, as has been the case up to now, there can be only a countable set of first-order formulas, since every formula is a finite string. Thus, for such L, every set T of axioms which has a model has a *countable* model, by 2.5.

PROOF OF 2.1. Let $\{\psi_1, \ldots, \psi_k\}$ be a finite set of first-order sentences true in all divisible abelian groups and let ψ be the conjunction $(\psi_1 \wedge \cdots \wedge \psi_k)$. Our task is to prove that ψ is true in some nondivisible abelian group. We apply the second version of the Compactness Theorem. Let T be the set of axioms (1)–(4) plus all the axioms $(6)_n$. Thus T is a set of axioms for divisible abelian groups. The hypothesis is that $T \models \psi$. By the Compactness Theorem there is a finite $T_0 \subseteq T$ such that $T_0 \models \psi$. This means that there is an N such that ψ is true in all abelian groups which satisfy $\forall x\, \exists y\, [ny = x]$ for $n = 2, \ldots, N$. (This much of the proof is common to many proofs.) Taking the first example which comes to mind, let $\mathbb{Z}_p$ be the group of integers mod p, for some prime $p > N$. The group $\mathbb{Z}_p$ is a model of $\forall x\, \exists y\, [ny = x]$ for $n < p$, since the map which sends x to nx is one–one and hence onto. Thus ψ is true in $\mathbb{Z}_p$. But $\mathbb{Z}_p$ is far from being divisible since $px = 0$ for all $x \in \mathbb{Z}_p$. □

The proof of 2.2 is just like the proof of 2.1 in form so is left to the reader.

PROOF OF 2.3. Let $G = \langle G, +, 0\rangle$ be any (possibly torsion) group such that, for each n, there is an element x_n of G of order $\geq n$. For example, G might be the direct sum of all $\mathbb{Z}_p$ over all primes p. We will prove that there is a nontorsion group H such that G and H satisfy exactly the same first-order sentences. Again we use the Compactness Theorem, this time the first version. Take a new constant symbol c and let T consist of all sentences (not mentioning c) true in the group $\langle G, +, 0\rangle$ plus all the sentences: $2c \neq 0$, $3c \neq 0$, $4c \neq 0$, etc. Thus T is a set of sentences in a language which has a name c for a new distinguished element. If $H = \langle H, +, 0, c\rangle$ satisfies all the axioms in T then $\langle H, +, 0\rangle$ will be a group with the same first order axioms true as are true in G but H will not be torsion since the distinguished element c will have infinite order. All we need to see is that there is an H which is a model of all of T. By the Compactness Theorem, it suffices to find a model of each finite $T_0 \subseteq T$. But this is easy. Given T_0, let N be bigger than all n such that the sentence $nc \neq 0$ is in T_0. Then we can use x_N to make T_0 true. That is, T_0 is true in the group $\langle G, +, 0, x_N\rangle$ with distinguished element x_N, since the order of x_N is $\geq N$. □

The real numbers

Our first set of examples had to do with whole classes of structures. We now turn to one specific structure, the ordered field $\mathbb{R} = \langle \mathbb{R}, +, \cdot, <, 0, 1\rangle$ of real numbers. Most students of advanced calculus suffer through a construction of $\mathbb{R}$ and a proof that certain axioms characterize $\mathbb{R}$ up to isomorphism. The axioms are not first-order, however.

2.6. PROPOSITION. *There is no first-order set of axioms which characterize $\mathbb{R}$ up to isomorphism.*

PROOF. By the remark following 2.5, the set of all sentences true in ℝ is countable. By the Löwenheim–Skolem Theorem, this set has a countable model. □

Since the Löwenheim–Skolem Theorem also holds for weak second-order logic, the proof of 2.6 shows that there is a countable field with the same weak second-order properties as ℝ, among which is the Archimedean axiom:

$$\forall x\, \exists n\, [x \leq n1],$$

where $n1$ is the term $((1+1)+\cdots+1)$, n-times, as before. This is a weak second-order statement since the quantifier $\exists n$ ranges not over the elements of an arbitrary model but over the real natural numbers.

The proof of 2.6 is misleading because it makes one feel that the problem has to do with the fact that there are undefinable real numbers, since there are more reals than there are possible definitions in first-order logic with countably many symbols. We can correct this impression by proving a similar result for the enriched structure $\langle \mathbb{R}, +, \cdot, <, r\rangle_{r\in\mathbb{R}}$ where *every* real number r is treated as a distinguished element and is given a name (i.e. constant symbol). We continue the (slightly confusing) practice of using an object r for its own name.

2.7. PROPOSITION. *There is a non-Archimedean field* $^*\mathbb{R}$ *extending* ℝ *which satisfies all the first-order sentences true in* ℝ, *even if we allow names for all real numbers.*

PROOF. The proof is similar, but actually simpler than, the proof of 2.3. We take another new constant symbol c and write the sentence

$$c > r$$

for all real numbers r. To these sentences we add all true first-order sentences of ℝ. By the Compactness Theorem this set of sentences has a model $^*\mathbb{R}$. We can consider ℝ as a submodel of $^*\mathbb{R}$. Since the field axioms are true in ℝ they are also true in $^*\mathbb{R}$. □

Most of the theorems of calculus are first-order so that they will hold in $^*\mathbb{R}$. Thus 2.5, far from being a negative result, is actually the basis of analysis by means of infinitesimals, or, in other words, Robinson's "nonstandard" analysis. (The element $1/c$ will be a positive infinitesimal.) Thus, it is only a mild exaggeration to say that the universal symbolic calculus of Leibniz' imagination eventually led to a justification of his use of infinitesimals in the calculus. For more on this, see Chapter A.6 and its 1.1 in particular.

Proposition 2.7 leaves us with the question: Which of the usual axioms for the real numbers is not first-order? The answer is: the completeness axiom,

$$\forall X \subseteq \mathbb{R}\, [\text{if } X \neq \emptyset \text{ is bounded, then } X \text{ has a l.u.b.}].$$

This is not first-order because the universal quantifier ranges over the set of all subsets X of ℝ. Thus, the proof that the real numbers are unique is really *relative* to a universe of set theory.

Rings and fields

The completeness property of the field of real numbers is not first-order, as we have seen. Let us conclude this introduction into first-order properties by seeing some of the properties of rings and fields that *are* first order. In this discussion our basic language (or vocabulary) L has the nonlogical symbols $+, \cdot, 0, 1$. The basic axioms for *commutative rings* with identity consist of (1)–(4) above (the abelian group axioms) plus the following first-order axioms:

$$\forall x\, \forall y\, [x \cdot y = y \cdot x],$$

$$\forall x\, \forall y\, \forall z\, [(x \cdot y) \cdot z = x \cdot (y \cdot z)],$$

$$\forall x\, \forall y\, \forall z\, [x \cdot (y + z) = (x \cdot y) + (x \cdot z)],$$

$$\forall x\, [x \cdot 1 = x],$$

$$0 \neq 1.$$

A ring $\mathfrak{R} = \langle R, +, \cdot, 0, 1\rangle$ is an *integral domain* if it is a model of

$$\forall x\, \forall y\, [x \cdot y = 0 \rightarrow (x = 0 \vee y = 0)].$$

Before going on to fields, let us pause to see what to do about a prime concern in ring theory, the notion of an ideal. A proper *ideal* of a commutative ring $\mathfrak{R} = \langle R, +, \cdot, 0, 1\rangle$ is simply a nonempty, proper subset $I \subseteq R$ which is a subgroup of $\mathfrak{R}$ under addition such that for all $x \in R$ and all $y \in I$, $x \cdot y \in I$. To express this in first-order logic we add a name for I and consider structures of the form $(\mathfrak{R}, I) =_{\text{df}} \langle R, +, \cdot, 0, 1, I\rangle$. Then I is an ideal of $\mathfrak{R}$ if $(\mathfrak{R}, I)$ is a model of the following three axioms. To keep set theory out of the picture, we think of I as a 1-place relation and write $I(x)$ rather than $x \in I$. The middle two axioms assert that I is a subgroup under $+$; the last asserts that I is closed under multiplication by any element x of $\mathfrak{R}$.

$$I(0) \wedge \neg I(1),$$

$$\forall x \forall y [I(x) \wedge I(y) \rightarrow I(x+y)],$$

$$\forall x \forall y [I(x) \wedge x + y = 0 \rightarrow I(y)],$$

$$\forall x \forall y [I(y) \rightarrow I(x \cdot y)].$$

An ideal I is a *prime ideal* if $(\mathfrak{R}, I)$ is a model of

$$\forall x \forall y [I(x \cdot y) \rightarrow I(x) \vee I(y)].$$

Up till now everything has been simple. Either the natural definition of a notion was first-order, or else we have been able to show that the notion is not first-order. This is not always the case. Indeed, some of the most useful applications of logical tools (like ultraproducts) hinge on finding some first-order equivalent to a notion that doesn't look first-order. This is often a nontrivial matter but we give only a simple example. Others can be found in Chapters A.3 and A.4.

An ideal I is a *maximal ideal* of $\mathfrak{R}$ if $(\mathfrak{R}, I)$ is a model of

$$\forall J [I \subseteq J \wedge J \text{ an ideal} \rightarrow J = I \text{ or } J = R]. \tag{9}$$

This is the same form of second-order sentence as the completeness axiom for the reals, but this time we can find an equivalent first-order axiom by recalling the lemma which says that I is maximal in $\mathfrak{R}$ iff the quotient ring $\mathfrak{R}/I$ is a field. To say that $\mathfrak{R}/I$ is a field is to say that for all x, if $x + I$ is not the coset $0 + I$, then there is a y such that $(x+I)(y+I) = 1 + I$. Since $(x+I)\cdot(y+I) = x \cdot y + I$, we can express (9) by the axiom

$$\forall x [\neg I(x) \rightarrow \exists y (x \cdot y + I = 1 + I)]$$

which, when written out in detail becomes

$$\forall x [\neg I(x) \rightarrow \exists y \exists z (I(z) \wedge xy + z = 1)]. \tag{9'}$$

While (9)′ looses the intuitive content of (9), it is equivalent to (9) and it is first-order, which is what matters here.

Here is a good exercise. A ring $\mathfrak{R}$ is a *principal ideal ring* if $\mathfrak{R}$ is a model of the second-order sentence

$$\forall I [I \text{ an ideal} \rightarrow \exists x \forall y (I(y) \leftrightarrow \exists z (y = zx))].$$

This has the same general form as (9) but it cannot be expressed in first-order logic. Indeed, a simple compactness argument shows that there is a ring $\mathfrak{R}$ with the same first order properties as the ring $\mathbb{Z}$ of integers (written $\mathfrak{R} \equiv \mathbb{Z}$) but where $\mathfrak{R}$ is not a principal ideal ring.

A commutative ring $\mathfrak{R}$ is a *field* if $\mathfrak{R}$ is a model of

$$\forall x \exists y [x \neq 0 \rightarrow x \cdot y = 1].$$

A field R is of *characteristic p* (p a prime) if R is a model of

$$p1 = 0.$$

On the other hand R is of *characteristic* 0 if

$$\forall p [p \text{ a prime} \rightarrow p1 \neq 0]. \tag{10}$$

Did you catch the weak second-order sentence? The quantifier ranges not over $\mathfrak{R}$ but over the prime numbers. Thus we must replace (10) by an infinite list:

$$p1 \neq 0, \tag{10$_p$}$$

one axiom for each prime p. The result corresponding to Propositions 2.1 and 2.2 becomes more interesting here. The proof is just like the proof of 2.1.

2.8. PROPOSITION. *Any first-order sentence ψ true in all fields of characteristic 0 is true in all fields of characteristic p for sufficiently large p, that is, for p greater than some integer N_ψ.*

Let us abbreviate the formal term $(x \cdot x)$ by x^2 and, by induction, abbreviate the formal term $(x^n \cdot x)$ by x^{n+1}. A field R is *algebraically closed* if it is a model of all axioms of the form

$$\forall x_0 \cdots \forall x_n [x_n \neq 0 \rightarrow \exists y (x_n \cdot y^n + x_{n-1} y^{n-1} + \cdots + x_1 y + x_0 = 0)],$$

which says that every polynomial of degree n has a root.

Set theory

The first-order axioms for set theory, are discussed at length in Shoenfield. The basic language L of set theory has only a membership symbol $\in$. The axioms are arrived at by a careful analysis of our informal concept of forming sets, sets of sets, sets of sets of sets, and so on into the transfinite. The resulting set of axioms is called ZF, after Zermelo and Fraenkel. The first axiom about sets one thinks of is the axiom of extensionality: a set is completely determined by its members. This becomes

$$\forall x \forall y [\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y].$$

Properties of mathematical theories

The various first-order theories we have discussed above have radically different properties from a logical point of view. Let us mention a few of them.

The theory of abelian groups is a *decidable theory*, whereas the theory of groups is undecidable. That is, one can give an effective procedure which will tell of an arbitrary sentence ψ involving $+$ and 0 whether or not ψ is a logical consequence of (1)–(4), i.e., whether or not ψ is true in all abelian groups. There can be no such procedure for the theory of groups. This sort of question is dealt with in Chapter C.1 and, more fully, in Chapter C.3.

The theory of algebraically closed fields of a fixed characteristic is a *complete theory*, which is to say that any two algebraically closed fields F_1, F_2 of the same characteristic have all the same first-order properties, i.e., $F_1 \equiv F_2$. On the other hand, most of the first-order theories are not complete. For example, to the theory of rings we can add either $\forall x\, \exists y\, [x \neq 0 \rightarrow x \cdot y = 1]$ or its negation $\neg \forall x\, \exists y\, [x \neq 0 \rightarrow x \cdot y = 1]$ and have a *consistent theory*. This just amounts to the triviality that some rings are fields and some are not. Combining the above mentioned completeness of the theory of algebraically closed fields with the Completeness Theorem shows, by Theorem 7.2 in Chapter C.1, that the theory of algebraically closed fields of characteristic 0 is *decidable*. Consider the effective procedure P for deciding whether or not a sentence involving $+, \cdot, 0, 1$ is a consequence of this theory. Since all models of this theory have the same first-order properties, we can apply P to decide which sentences involving $+, \cdot, 0, 1$ are true in the field $\mathbb{C} = \langle \mathbb{C}, +, \cdot, 0, 1 \rangle$ of complex numbers. This is expressed by saying that the field $\mathbb{C}$ of complex numbers is a *decidable model*.

Gödel's famous Incompleteness Theorem shows that the ring $\mathbb{Z}$ of integers is *not decidable*. Thus, any mechanical procedure which attempts to decide of sentences ψ involving $+, \cdot, 0, 1$ whether or not ψ is true in $\mathbb{Z}$ must fail for infinitely many sentences. A consequence of this is that any effective list T of true axioms we write down about $\mathbb{Z}$ must inveitably yield an incomplete theory (since otherwise the argument used on $\mathbb{C}$ would work on $\mathbb{Z}$). Gödel's Second Incompleteness Theorem in fact tells us how to go about finding a sentence ψ true in $\mathbb{Z}$ but not a consequence of T. Chapter D.1 contains a thorough discussion of Gödel's Incompleteness Theorems. These results are usually stated in terms of the structure $\mathbb{N} = \langle \mathbb{N}, +, \cdot, 0, 1 \rangle$ of natural numbers, rather than in terms of the ring $\mathbb{Z}$. The standard definition of $\mathbb{Z}$ from $\mathbb{N}$ shows that the results apply equally to $\mathbb{Z}$.

There are a number of topics that could be gone into at this point, but it is more reasonable to let the topics speak for themselves in the chapters that follow. Chapter A.2 discusses the basics of the theory of models for first-order logic. Chapter A.3 treats the ultraproduct construction, an algebraic version of the compactness theorem. Chapter A.4 will also be of particular interest to algebraists, treating as it does, model theoretic analogues of the notion of "algebraically closed" and their applications in algebra. The fundamental results of Ax–Kochen and Ersov are discussed in both of the chapters.

3. The formalization of first-order logic

Let L be a given set of function symbols, relation symbols and constant symbols. We make no restriction on the size of the set L, though usually L is finite or countably infinite. Each function symbol $f \in \mathrm{L}$ has a positive integer $\#(f)$ assigned to it; if $n = \#(f)$, then f is called an n-ary function symbol. Similarly, each relation symbol $R \in \mathrm{L}$ comes with a positive integer $\#(R)$; if $n = \#(R)$ then R is said to be an n-ary relation symbol.

Examples. For the language $\mathrm{L} = \{+, 0\}$ appropriate to group theory there are no relation symbols and $\#(f) = 2$. For the language $\mathrm{L} = \{\in\}$ of set theory, there are no functions or constant symbols and $\#(\in) = 2$.

Given a language L we have a natural notion of *structure* or *model* for L. A structure $\mathfrak{M}$ assigns a nonempty collection M of objects over which the quantifiers range, and $\mathfrak{M}$ also assigns appropriate interpretations of the basic primitive relation, function and constant symbols of L.

3.1. Definition. A (*set-theoretic*) *structure for* L is a pair $\mathfrak{M} = \langle M, F \rangle$ where M is a nonempty set and F is an operation with domain L such that, writing $x^{\mathfrak{M}}$ for $F(x)$,

(i) if $R \in \mathrm{L}$ is an n-ary relation symbol, then $R^{\mathfrak{M}} \subseteq M^n$;
(ii) if $f \in \mathrm{L}$ is an n-ary function symbol, then $f^{\mathfrak{M}} : M^n \rightarrow M$;
(iii) if $c \in \mathrm{L}$ is a constant symbol then $c^{\mathfrak{M}} \in M$.

One often writes $\mathfrak{M}$ as $\langle M, R^{\mathfrak{M}}, \ldots, f^{\mathfrak{M}}, \ldots, c^{\mathfrak{M}}, \ldots \rangle$. The parenthetical adjective "set-theoretic" in 3.1 is there because one sometimes wants to consider more generous notions of structures where M may be too large to be a set. For example, the natural structue $\mathfrak{M}$ for the language $\mathrm{L} = \{\in\}$ of set theory has domain M the collection V of all sets, which is not itself a set. However, a consequence of the Completeness Theorem is that any set of axioms that has a model in any reasonable sense will have as a model a reasonably small set theoretic structure. We henceforth deal only with set-theoretic structures.

Example. If $L = \{+, 0\}$ is the language appropriate to group theory, then a structure for L has the form $\mathfrak{M} = \langle M, +^M, 0^M \rangle$ where M is a nonempty set, $+^M : M \times M \to M$ and $0^M \in M$. We usually use G rather than $\mathfrak{M}$ and drop the superscripts.

We now turn to syntactic notions of first-order logic. Recall the basic building blocks $\wedge, \vee, \neg, \to, =, \forall, \exists, x, y, z, \ldots,), ($, mentioned early in Section 2. Let L be a fixed language. Any finite sequence, each element of which is one of these basic symbols or an element of L, is called an *expression*. From the set of expressions we want to single out the ones to which we can assign a meaning.

3.2. DEFINITION. The *terms* of L form the smallest set of expressions containing the variables $x, y, z, \ldots$, all constant symbols of L (if any) and closed under the formation rule: if $t_1, \ldots, t_n$ are terms of L and if $f \in L$ is an n-ary function symbol, then the expression $f(t_1 \cdots t_n)$ is a term of L. A *closed term* is a term in which no variable appears.

If there are no function symbols in L then the formation rule is vacuous so the only terms are variables and the constants of L.

Example. If $L = \{+, 0\}$ then, strictly speaking, the terms are expressions like

$$+(xy), \qquad +(0 + (x0)).$$

We naturally agree to abbreviate these by the more natural

$$x + y, \qquad 0 + (x + 0),$$

respectively, thus moving the symbol $+$ inside and leaving off the outer parentheses if no confusion arises. As in Section 2 we use nx as an abbreviation of $(\cdots((x + x) + x) + \cdots + x)$, n times, for $n \geq 1$. For this language the only closed terms are the expressions built up from 0 and $+$, none of which are very interesting from a group theoretic point of view.

3.3. DEFINITION. An *atomic formula* of L is an expression of either of the two forms:

$$(t_1 = t_2), \qquad R(t_1 \ldots t_n)$$

where, in the first case, t_1 and t_2 are terms of L. In the second case $R \in L$ is any n-ary relation symbol and $t_1, \ldots, t_n$ are terms of L.

Examples. In the language $L = \{+, 0\}$ of group theory there are not any relation symbols, so the only atomic formulas are statements of equalities between terms, expressions like

$$(x + y = z), \qquad (x + y = y + x), \qquad (x + y) + z = x + (y + z).$$

In the language $L = \{\in\}$ of set theory where all terms are variables, the only atomic formulas are those of the form $(v = w)$ and $\in(vw)$ for variables v, w. We write the latter as $v \in w$.

3.4. DEFINITION. The first-order *formulas* of L form the smallest set of expressions containing the atomic formulas and closed under the following formation rules:

(i) If φ, ψ are formulas so are the expressions

$$\neg\varphi, \quad (\varphi \wedge \psi), \quad (\varphi \vee \psi), \quad (\varphi \to \psi);$$

(ii) if φ is a formula and v is a variable, then $(\exists v \varphi)$ and $(\forall v \varphi)$ are formulas.

We associate parentheses to the right in strings where the same symbol is repeated. Thus $\varphi \wedge \psi \wedge \theta$ is $(\varphi \wedge (\psi \wedge \theta))$ and $\varphi \to \psi \to \theta$ is $(\varphi \to (\psi \to \theta))$.

Example. Let $L = \{+, 0\}$. The following are formulas:

$$(x + y = 0),$$

$$(\exists y\,(x + y = 0)),$$

$$(\forall x\,(\exists y\,(x + y = 0)))$$

The last is what we wrote more informally as sentence (3) in Section 2. Note that in the first formula both x and y are sort of "floating free", in the second formula y is "bound up" by $\exists$ and in the last formula both x and y are "bound". Only the last formula makes any intuitive sense as an axiom. This is similar to the situation in elementary calculus where

$$x^2 + 2x + 1$$

is an expression which has a variable in it, but the expression

$$\int_0^1 (x^2 + 2x + 1)\,dx$$

has x "bound"; it has a meaning independent of x. The definite integral is performing roughly the same syntactic role that the quantifiers $\exists$ and $\forall$ play in logic. The next definition makes the notion of "free variable" precise. One can think of it as defined by induction on the length of the formula φ.

3.5. Definition. The set $FV(\varphi)$ of *free variables* of a formula φ is defined as follows:

(i) If φ is an atomic formula, then $FV(\varphi)$ is just the set of variables appearing in the expression φ,

(ii) $FV(\neg\varphi) = FV(\varphi)$,

(iii) $FV(\varphi \wedge \psi) = FV(\varphi \vee \psi) = FV(\varphi \to \psi) = FV(\varphi) \cup FV(\psi)$,

(iv) $FV(\exists v \varphi) = FV(\forall v \varphi) = FV(\varphi) - \{v\}$.

It is common practise to use the notation $\varphi(v_1, \ldots, v_n)$ to indicate that $FV(\varphi) \subseteq \{v_1, \ldots, v_n\}$ without implying that all of $v_1 \cdots v_n$ are actually free in φ. This is similar to the practise in algebra of writing $p(x_1, \ldots, x_n)$ for a polynomial p in the variables $x_1, \ldots, x_n$ without implying that all of them have nonzero coeficient.

3.6. Definition. A (*first-order*) *sentence* of L is a formula without any free variables.

So far the terms, formulas and sentences of L are simply finite strings of symbols. We must make sure to assign the intended meanings to our logical symbols so that the formulas of Section 2 express what we intend. This is done by defining the *satisfaction relation* $\mathfrak{M} \models \varphi$ between structures on the one hand (the left one) and sentences on the other.

Let $\mathfrak{M} = \langle M, \ldots \rangle$ be a structure for a language L. An *assignment in* $\mathfrak{M}$ is a function s with domain the set of variables of L and range a subset of M. We think of s as assigning a meaning $s(v)$ to the variable v. We can then define, for each term t of L a function $t^{\mathfrak{M}}$ which maps assignments to elements of M.

3.7. Definition. Let M be given. For t a term of L define $t^{\mathfrak{M}}$ as follows:

(i) If t is a constant symbol c, then $t^{\mathfrak{M}}(s) = c^{\mathfrak{M}}$ for all s;

(ii) if t is a variable v, then $t^{\mathfrak{M}}(s) = s(v)$ for all s;

(iii) if t is the term $f(t_1 \cdots t_n)$ then, for all s, define

$$t^{\mathfrak{M}}(s) = f^{\mathfrak{M}}(t_1^{\mathfrak{M}}(s), \ldots, t_n^{\mathfrak{M}}(s)).$$

In (iii), since each of $t_1, \ldots, t_n$ is simpler than t we can assume by induction on (the complexity of) terms that $t_1^{\mathfrak{M}}, \ldots, t_n^{\mathfrak{M}}$ are already defined. $f^{\mathfrak{M}}$ is defined since $\mathfrak{M}$ is a structure for L and $f \in$ L. The reader should note that if $s_1(v) = s_2(v)$ agree on all variables v appearing in t, then $t^{\mathfrak{M}}(s_1) = t^{\mathfrak{M}}(s_2)$. Thus $t^{\mathfrak{M}}$, as a function, depends on only a finite number of values of its argument s.

Example. Let L be the language of rings and let t be the term, or polynomial,

$$x^2 + 2x + 1.$$

Then $t^{\mathfrak{R}}$, for any ring $\mathfrak{R}$, is the corresponding *polynomial function* from $\mathfrak{R}$ into $\mathfrak{R}$. If $s(x) = a$, then $t^{\mathfrak{R}}(s) = a^2 + 2a + 1$, the operations of + and · being those of the ring $\mathfrak{R}$.

In the following definition we use $s(\substack{a \\ v})$ for the assignment s' which agrees with s except that $s'(v) = a$.

3.8. Definition. Let $\mathfrak{M}$ be an L-structure. We define a relation

$$\mathfrak{M} \models \varphi[s],$$

(read: the assignment s satisfies the formula φ in $\mathfrak{M}$) for all assignments s and all formulas φ as follows.

(i) $\mathfrak{M} \models (t_1 = t_2)[s]$ iff $t_1^{\mathfrak{M}}(s) = t_2^{\mathfrak{M}}(s)$,

(ii) $\mathfrak{M} \models R(t_1 \cdots t_n)[s]$ iff $(t_1^{\mathfrak{M}}(s), \ldots, t_n^{\mathfrak{M}}(s)) \in R^{\mathfrak{M}}$,

(iii) $\mathfrak{M} \models \neg\varphi[s]$ iff not $\mathfrak{M} \models \varphi[s]$,

(iv) $\mathfrak{M} \models (\varphi \wedge \psi)[s]$ iff $\mathfrak{M} \models \varphi[s]$ and $\mathfrak{M} \models \psi[s]$,

(v) $\mathfrak{M} \models (\varphi \vee \psi)[s]$ iff $\mathfrak{M} \models \varphi[s]$ or $\mathfrak{M} \models \psi[s]$ or both,

(vi) $\mathfrak{M} \models (\varphi \to \psi)[s]$ iff either not $\mathfrak{M} \models \varphi[s]$ or else $\mathfrak{M} \models \psi[s]$,

(vii) $\mathfrak{M} \models (\exists v \varphi)[s]$ iff there is an $a \in M$ such that $\mathfrak{M} \models \varphi[s(\substack{a \\ v})]$,

(viii) $\mathfrak{M} \models (\forall v \varphi)[s]$ iff for all $a \in M$, $\mathfrak{M} \models \varphi[s(\substack{a \\ v})]$.

There is nothing surprising here. It is just making sure that each of ou symbols means what we want it to mean. There is one possibly confusin point, in (i), caused by our using = for both the real equality (on th right-hand side) and the symbol for equality (on the left). Many author abhor this confusion of use and mention and use something like $\equiv$ or $\approx$ for the symbol.

The reader should observe that the truth or falsity of $\mathfrak{M} \models \varphi[s]$ depend only on the values of $s(v)$ for variables v which are actually free in φ. Tha is, if $s_1(v) = s_2(v)$ for all v free in φ, then $\mathfrak{M} \models \varphi[s_1]$ iff $\mathfrak{M} \models \varphi[s_2]$. Thus, if is $\varphi(v_1 \cdots v_n)$ and $a_1 = s(v_1), \ldots, a_n = s(v_n)$, then we may writ

$\mathfrak{M} \models \varphi[a_1, \ldots, a_n]$ for $\mathfrak{M} \models \varphi[s]$ without confusion. Also, if φ is a sentence, then the truth or falsity of $\mathfrak{M} \models \varphi[s]$ is completely independent of s. Thus we may write $\mathfrak{M} \models \varphi$ (read: $\mathfrak{M}$ is a model of φ; or $\mathfrak{M}$ satisfies φ) if for some (hence every) assignment s, $\mathfrak{M} \models \varphi[s]$.

If $\varphi(v)$ is a formula and t is a term then $\varphi(t/v)$ (or, more simply, $\varphi(t)$) denotes the result of replacing all occurrences of the free variable v by the term t throughout. When using this notation we always assume that none of the variables in t occur as bound variables in φ. If they did we could always rename the bound variables. Otherwise we would distort the meaning of $\varphi(t)$. For example, if t is w and $\varphi(v)$ is $\exists w\, (v \neq w)$, then $\varphi(t)$ should assert $\exists w'\, (w \neq w')$, not $\exists w\, (w \neq w)$

A structure $\mathfrak{M}$ is a *model of a set* Φ of sentences if $\mathfrak{M} \models \varphi$ for all $\varphi \in \Phi$. Given two structure $\mathfrak{M}$, $\mathfrak{N}$ for L, we say that $\mathfrak{M}$ and $\mathfrak{N}$ are *elementarily equivalent*, and write $\mathfrak{M} \equiv \mathfrak{N}$, iff for all sentences φ of L, $\mathfrak{M} \models \varphi$ iff $\mathfrak{N} \models \varphi$. If $\mathfrak{M} \cong \mathfrak{N}$ (i.e. $\mathfrak{M}$ is isomorphic to $\mathfrak{N}$, in the obvious sense) then $\mathfrak{M} \equiv \mathfrak{N}$. Finally, let $\mathcal{K}$ be a class of structures for a language L. $\mathcal{K}$ is *(finitely) axiomatizable* if there is a (finite) set Φ of first-order sentences of L such that, for all structures $\mathfrak{M}$, $\mathfrak{M} \in \mathcal{K}$ iff $\mathfrak{M}$ is a model of Φ. This agrees with our terminology in Section 2. Some authors call a finitely axiomatizable class an *elementary* class, or EC. They are then forced into calling an axiomatizable class *elemenatary in the wider sense*, or EC_Δ

4. The Completeness Theorem

Surely the most important discovery for mathematics by the ancient Greeks was of the notion of *proof*, turning mathematics into a deductive science. Each theorem φ must have a proof from a set T of more or less explicitly stated assumptions, or *axioms*. The proof must demonstrate that the conclusion φ follows from the axioms in T by the laws of logic alone. The mathematician implictly assumes that he understands the notion of proof and that, in particular, he will be able to check in a rigorous manner whether a purported complete proof does indeed establish the conclusion from the stated assumptions. The natural question is: Can the notions "laws of logic" and "proof" be made mathematically precise?

In this section we want to show that there is a mathematically precise notion of "φ is provable from T" which captures completely the intuitive notion "φ follows from T by the laws of logic alone", for first-order φ and T. More fully, we want to provide a concrete set of obviously valid rules of inference such that φ follows from T by the laws of logic alone if and only

if there is a proof of φ from axioms in T which uses only the permitted rules of inference.

There is a seeming obstacle to our program. How can we hope to prove such a result without knowing in advance what it means to follow by the laws of logic alone? Luckily, we do not need to know. All we need is to agree that, whatever it means, it at least *implies* that φ will hold in all set theoretic structures which are models of T; i.e., it implies $T \models \varphi$. Thus, to realize our goal, it more than suffices to provide valid rules of inference and show that $T \models \varphi$ if and only if φ is provable from T. This is the content of Gödel's Completeness Theorem.

The plan of this section is as follows. In 4.1 and 4.2 we take care of so-called propositional logic. In 4.3–4.8 we discuss a method, due in essence to Henkin, for reducing certain problems of first-order logic back to problems about propositional logic. The proofs of the Compactness Theorem and the Löwenheim–Skolem Theorem fall out of this method. Finally we present two different versions of the Gödel Completeness Theorem which are consequences of 4.8., a Hilbert-style formal system (4.9) and a Gentzen-style formal system (4.13).

Propositional logic

It is expeditious to break the study of first-order logic up into two parts, the trivial part having to do with the propositional connectives $\wedge, \vee, \neg, \rightarrow$, and then the part having to do with equality and the quantifiers $\forall$ and $\exists$.

Let P be a set of objects called *prime formulas*. They might be sentences of some natural language or letters $p, q, r, \ldots$ of the alphabet, for example. In our application, they will be those first-order formulas which are not propositional combinations of simpler formulas, that is, atomic formulas and formulas beginning with a quantifier. The set of *propositional formulas* of P form the smallest set of expressions containing the members of P and closed under the rule: if A, B are propositional formulas then so are $\neg A$, $(A \wedge B)$, $(A \vee B)$ and $(A \rightarrow B)$. The *prime constituents* of a propositional formula A are just the prime formulas out of which A is built.

Examples. Suppose $P = \{p, q, r\}$. The following are propositional formulas of P:

$$p, \quad q, \quad (p \vee q), \quad (q \vee p), \quad ((p \vee q) \rightarrow (q \vee p)).$$

We want to show exactly how the truth or falsity of a propositional formula depends on the truth or falsity of its prime constituents. Then, going a step further, we show how to decide which propositional formulas are always true, regardless of the truth or falsity of their prime constituents, formulas like $(p \vee \neg p)$, $(((p \rightarrow q) \wedge \neg q) \rightarrow \neg p)$, etc. Such formulas are called propositional *tautologies*, since they are true by virtue of their syntactic form alone. These tautologies provide a small first step in isolating the laws of logic.

Let **t** and **f** be distinct new symbols, thought of as "true" and "false". A *truth assignment* for a set P of prime formulas is, by definition, a function $\nu : P \rightarrow \{\mathbf{t}, \mathbf{f}\}$. For each truth assignment ν we define its extension $\bar{\nu}$ to the set of all propositional formulas of P by induction on length of formulas as follows:

$$\bar{\nu}(A) = \nu(A) \quad \text{if } A \text{ is prime;}$$

$$\bar{\nu}(\neg A) = \mathbf{f} \quad \text{if } \bar{\nu}(A) = \mathbf{t},$$
$$= \mathbf{t} \quad \text{if } \bar{\nu}(A) = \mathbf{f};$$

$$\bar{\nu}(A \wedge B) = \mathbf{t} \quad \text{if } \bar{\nu}(A) = \bar{\nu}(B) = \mathbf{t},$$
$$= \mathbf{f} \quad \text{otherwise;}$$

$$\bar{\nu}(A \vee B) = \mathbf{t} \quad \text{if } \bar{\nu}(A) = \mathbf{t} \text{ or } \bar{\nu}(B) = \mathbf{t} \text{ or both,}$$
$$= \mathbf{f} \quad \text{otherwise;}$$

$$\bar{\nu}(A \rightarrow B) = \mathbf{f} \quad \text{if } \bar{\nu}(A) = \mathbf{t} \text{ and } \bar{\nu}(B) = \mathbf{f},$$
$$= \mathbf{t} \quad \text{otherwise.}$$

This definition can be summarized by means of the following *truth table*:

A	B	$\neg A$	$(A \wedge B)$	$(A \vee B)$	$(A \rightarrow B)$
t	**t**	**f**	**t**	**t**	**t**
t	**f**	**f**	**f**	**t**	**f**
f	**t**	**t**	**f**	**t**	**t**
f	**f**	**t**	**f**	**f**	**t**

By constructing such truth tables we can completely analyze how the truth or falsity of a propositional formula depends on the truth or falsity of its prime constituents. We illustrate the method for the formula

$$((\neg(p \wedge \neg q) \wedge q) \rightarrow p).$$

We simplify the table by leaving out some of the **t**'s.

p	q	$\neg q$	$p \wedge \neg q$	$\neg(p \wedge \neg q)$	$\neg(p \wedge \neg q) \wedge q$	$(\neg(p \wedge \neg q) \wedge q) \rightarrow$
t	**t**	**f**	**f**			
t	**f**			**f**	**f**	
f	**t**	**f**	**f**			**f**
f	**f**		**f**		**f**	

Thus, the only circumstances under which our final formula is false is whe p is false and q is true.

4.1. Definition. A propsoitional formula A of P is a *tautology* if $\bar{\nu}(A) =$ for all truth assignments $\nu : P \rightarrow \{\mathbf{t}, \mathbf{f}\}$. A is *consistent* if $\bar{\nu}(A) = \mathbf{t}$ for som $\nu : P \rightarrow \{\mathbf{t}, \mathbf{f}\}$.

The method of truth tables makes it a trivial matter to see whether propositional formula is a tautology or not, or whether it is consistent o not. If we write $A \leftrightarrow B$ for $(A \rightarrow B) \wedge (B \rightarrow A)$, then we see that th following are tautologies:

$(A \vee \neg A)$ (law of the excluded middle),

$\neg(A \wedge \neg A)$ (law of contradiction),

$\left.\begin{array}{l}\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)\\ \neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)\end{array}\right\}$ (de Morgan's laws),

$\neg\neg A \leftrightarrow A$ (law of double negation).

Just to make sure the method of truth tables is perfectly clear, we presen an example with three prime constituents p, q, r:

$$\underbrace{[((p \wedge q) \rightarrow r)}_{A} \wedge \underbrace{(\neg r \rightarrow q)]}_{B} \rightarrow \underbrace{(p \rightarrow r)}_{C}$$

p	q	r	$p \wedge q$	A	$\neg r$	B	$A \wedge B$	C	$[A \wedge B] \rightarrow C$
t	**t**	**t**			**f**				
t	**t**	**f**		**f**			**f**	**f**	
t	**f**	**t**	**f**		**f**				
t	**f**	**f**	**f**			**f**	**f**	**f**	
f	**t**	**t**	**f**		**f**				
f	**t**	**f**	**f**						
f	**f**	**t**	**f**		**f**				
f	**f**	**f**	**f**			**f**	**f**		

Thus, since no falses turn up in the last column, the formula is indeed a tautology.

In practise, there is a much shorter method to check to see whether a formula is or is not a tautology. One works backwards, trying to find a consistent assignment which makes the formula false. Applied to the above, to make $[A \wedge B] \rightarrow C$ false, we need to have $\bar{\nu}(A) = \bar{\nu}(B) = \mathbf{t}$ but $\bar{\nu}(C) = \mathbf{f}$. To make $\bar{\nu}(C) = \mathbf{f}$, we must make $\bar{\nu}(p) = \mathbf{t}$, $\bar{\nu}(r) = \mathbf{f}$. To have $\bar{\nu}(B) = \mathbf{t}$ we must have $\bar{\nu}(q) = \mathbf{t}$, since $\bar{\nu}(\neg r) = \mathbf{t}$. But now we have $\bar{\nu}(p) = \bar{\nu}(q) = \mathbf{t}$ and $\bar{\nu}(r) = \mathbf{f}$ which gives $\bar{\nu}(A) = \mathbf{f}$, a contradiction. Thus the above formula is a tautology.

A set T of propositional formulas is said to be *consistent* (in the sense of propositional logic) if there is a truth assignment ν such that $\bar{\nu}(A) = \mathbf{t}$ for all $A \in T$.

4.2. Compactness Theorem for Propositional Logic. *A set T of propositional formulas is consistent if and only if every finite subset of T is consistent.*

Proof. We present two proofs of the nontrivial half.

First proof. For the purposes of this proof call a set S *finitely consistent* if every finite subset of S is consistent. We wish to prove that every finitely consistent set is consistent. Call S *maximal* finitely consistent if S is finitely consistent and for every formula A, either $A \in S$ or $(\neg A) \in S$.

There is a natural correspondence between valuations ν and maximal, finitely consistent sets. To any ν assign the set $S_\nu = \{A \mid \bar{\nu}(A) = \mathbf{t}\}$. This set is maximal, finitely consistent. Conversely, given a maximal, finitely consistent set S, define $\nu(p) = \mathbf{t}$ if $p \in S$, $\nu(p) = \mathbf{f}$ if $p \notin S$. The following facts follows immediately from the fact that S is maximal, finitely consistent, and imply (by induction on formulas A) that $S = S_\nu$:

$$B \in S \quad \text{iff} \quad (\neg B) \notin S,$$

$$(A \wedge B) \in S \quad \text{iff} \quad A \in S \quad \text{and} \quad B \in S,$$

$$(A \vee B) \in S \quad \text{iff} \quad A \in S \quad \text{or} \quad B \in S,$$

$$(A \rightarrow B) \in S \quad \text{iff} \quad A \notin S \quad \text{or} \quad B \in S.$$

For example, let's prove that $(A \vee B) \in S$ implies $A \in S$ or $B \in S$. Suppose not. Then $(A \vee B) \in S$ but $(\neg A) \in S$ and $(\neg B) \in S$, by maximality. But then $\{(A \vee B), \neg A, \neg B\}$ is a finite, inconsistent subset of S, a contradiction.

The above remarks show that proving a finitely consistent set T consistent is equivalent to finding a maximal, finitely consistent set $S \supseteq T$.

We show how to construct such an S in the case where the underlying set P of prime formulas is countable. Essentially the same proof works as long as P is well-ordered and hence, by the axiom of choice, works for all P. The proof for well-ordered P does not need the axiom of choice.

If P is countable, so is the set of all formulas so we enumerate them: $A_1, A_2, \ldots, A_n, \ldots$. Define $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_n \subseteq \cdots$ by

$$T_0 = T;$$

$$T_{n+1} = T_n \cup \{A_n\} \quad \text{if this is finitely consistent,}$$

$$= T_n \cup \{\neg A_n\} \quad \text{otherwise.}$$

Let $S = \bigcup T_n$. Clearly $T \subseteq S$ and for every A, either $A \in S$ or $(\neg A) \in S$. To finish the proof we need only show that each T_n, and hence S, is finitely consistent. This is proved by induction on n with $n = 0$ being the hypothesis of the theorem that T is finitely consistent. Assume T_n is finitely consistent, and prove that T_{n+1} is finitely consistent.

Case 1. $T_{n+1} = T_n \cup \{A_n\}$. By the definition of T_{n+1}, this is finitely consistent.

Case 2. $T_{n+1} = T_n \cup \{\neg A_n\}$. This can only happen if there is some finite set $T'_n \subset T_n$ such that $T'_n \cup \{A_n\}$ is not consistent. Suppose that T_{n+1} is not finitely consistent. Then there is a finite set $T''_n \subseteq T_n$ such that $T''_n \cup \{\neg A_n\}$ is not consistent. But then $T'_n \cup T''_n$ is a finite subset of T_n so is consistent. Any assignment ν making all of $T'_n \cup T''_n$ true must make *one* of A_n or $\neg A_n$ true, contradicting the inconsistency of both $T'_n \cup \{A_n\}$ and $T''_n \cup \{\neg A_n\}$.

Thus, in either case, T_{n+1} is after all finitely consistent. This finishes the proof.

Second proof. We can give a faster proof by quoting the Tychonoff Theorem. It hides the basic construction, though, and thus is less suitable for other constructions in model theory. Let $2 = \{\mathbf{t}, \mathbf{f}\}$ be the two element space with the discrete topology and let $X = 2^P$, the space of all truth assignments of P with the product topology. By the Tychonoff Theorem X is a compact, Hausdorff space. Hence if $\mathcal{F} = \{F_i \mid i \in I\}$ is an indexed family of closed subsets, and if $\bigcap_{i \in I} F_i = \emptyset$, then there is a finite $I_0 \subseteq I$ such that $\bigcap_{i \in I_0} F_i = \emptyset$. For each propositional formula A, let $F_A = \{\nu \in X \mid \bar{\nu}(A) = \mathbf{t}\}$. We claim that each F_A is clopen (both closed and open) in X. For $A = p$ a prime formula F_p is open by the very definition of the product topology. But $X - F_p = \{\nu \mid \nu(p) = \mathbf{f}\}$ is also open, by definition, so F_p is clopen. For more complicated formulas, the claim follows by induction on length of formulas and the following equations:

$$F_{(A\vee B)} = F_A \cup F_B, \qquad F_{(A\wedge B)} = F_A \cap F_B,$$
$$F_{(A\to B)} = F_B - F_A, \qquad F_{\neg A} = X - F_A.$$

This establishes the claim. Now let T be as given in the theorem. By hypothesis, for each finite $T_0 \subseteq T$, there is a $\bar{\nu}$ making all $\varphi \in T_0$ true, i.e. $\bigcap_{A\in T_0} F_A \neq \emptyset$. By the compactness of X, $\bigcap_{A\in T} F_A \neq \emptyset$. Thus, there is a truth assignment $\bar{\nu}$ making all $A \in T$ true. □

The standard classroom example of a simple application of the Compactness Theorem for Propositional Logic is to prove that if an infinite map cannot be colored by k colors then some finite submap cannot be colored by k colors. To prove this one assigns k prime formulas to each "country" on the map, one for each color, and writes down the obvious "axioms" asserting that each country gets exactly one color "true" and that adjacent countries do not have the same color "true". Another example, if one gives the first proof of the Compactness Theorem, is to prove the Tychonoff Theorem for 2^P.

The use of Henkin constants for reducing first-order logic to propositional logic

In this subsection we apply the notions of propositional logic to first-order logic. Given a language L let P be the set of formulas of L which are atomic or begin with ∀ or ∃. Thus, a *tautology of first-order logic* is any formula which is true regradless of what truth assignment is given to the prime formulas. For example the following are tautologies,

$$\forall x\, R(x) \vee \neg \forall x\, R(x),$$
$$\neg(\forall x\, R(x) \wedge \exists x\, S(x)) \leftrightarrow (\neg \forall x\, R(x) \vee \neg \exists x\, S(x)),$$

but the following sentences are not tautologies:

$$(c = c),$$
$$\forall x\, (R(x) \vee \neg R(x)),$$
$$\neg \exists x\, S(x) \to \forall x\, \neg S(x).$$

The first two are prime formulas, the third has the form $\neg p \to q$ for prime p and q. We see that the tautologies of first-order logic barely scratch the surface of the collection of "laws of logic".

We state an obvious lemma for the record.

4.3. Lemma. *Let $\mathfrak{M} = \langle M, \ldots \rangle$ be a structure for a language* L *and let s be an assignment in $\mathfrak{M}$, i.e., a function mapping the variables of* L *into M. There is a truth assignment ν to the prime formulas of* L *such that, for all formulas φ of* L, *$\mathfrak{M} \models \varphi[s]$ if and only if $\bar{\nu}(\varphi) = \mathbf{t}$. In particular, any set of sentences true in $\mathfrak{M}$ is consistent in the sense of propositional logic.*

Proof. For φ a prime formula, define $\nu(\varphi) = \mathbf{t}$ if $\mathfrak{M} \models \varphi[s]$, otherwise $\nu(\varphi) = \mathbf{f}$. Since every formula is built up from prime formulas by means of propositional connectives, the conclusion is obvious. □

The converse of the lemma is far from true. For example the following set of sentences is consistent in the sense of propositional logic (they are all prime formulas) but has no model:

$$\{\forall x\, (R(x) \to S(x)),\ \forall x\, R(x),\ \exists x\, \neg S(x)\}.$$

There has been no analysis of the quantificational structure of the sentences.

4.4. Equality axioms. The equality axioms are as follows, where $u, w, u_1, \ldots$ denote variables and constant symbols of L:

$$(u = u),$$
$$(u = w) \to (w = u),$$
$$(u_1 = u_2 \wedge u_2 = u_3) \to (u_1 = u_3),$$
$$(u_1 = w_1 \wedge \cdots \wedge u_n = w_n) \to (R(u_1 \cdots u_n) \to R(w_1 \cdots w_n)),$$
$$(u_1 = w_1 \wedge \cdots \wedge u_n = w_n) \to (t(u_1 \cdots u_n) = t(w_1 \cdots w_n)),$$

where R is an arbitrary n-ary relation symbol of L and t is an arbitrary n-ary term of L. The equality axioms are *valid* in that for all such axioms φ, all $\mathfrak{M}$ and all assignments s to variables, $\mathfrak{M} \models \varphi[s]$. The last four axioms for equality might well be called Leibniz Law.

The *witnessing expansion* L(C) of a language L is constructed as follows. Let $C_0 = \emptyset$ and, once C_n is defined, let $L_n = L \cup C_n$. For each formula $\varphi(v)$ of L_0 with exactly one free variable let $c_{\varphi(v)}$ be a distinct new constant symbol and let C_1 be the set of all these $c_{\varphi(v)}$. Given C_n, assign distinct new constant symbols $c_{\varphi(v)}$ to each formula $\varphi(v)$ of L_n which is not already a formula of L_{n-1} (i.e., if some constant from C_n appears in φ). Let C_{n+1} be C_n union the set of all these new $c_{\varphi(v)}$. Let $C = \bigcup_n C_n$ and let $L(C) = L \cup C$.

4.5. Definition. The constant symbol $c_{\varphi(v)}$ is called a *witnessing constant* and the sentences

I $\qquad (\exists v\, \varphi(v)) \rightarrow \varphi(c_{\varphi(v)}),$

II $\qquad \varphi(c_{\neg\varphi(v)}) \rightarrow \forall v\, \varphi(v)$

are called *Henkin axioms* of types I and II.

The informal idea behind the Henkin axioms is quite simple. If $\exists v\, \varphi(v)$ is true in a structure, choose an element a satisfying $\varphi(v)$ and give it a new name $c_{\varphi(v)}$. If $\forall v\, \varphi(v)$ is false, choose a counterexample b and call it by the new name $c_{\neg\varphi(v)}$.

4.6. Definition. T_{Henkin} is, by definition, the set of all sentences of L(C) which are either Henkin axioms or else of one of the forms:

III $\qquad \forall v\, \varphi(v) \rightarrow \varphi(t), \quad t$ a closed term of L(C);

IV $\qquad \varphi(t) \rightarrow \exists v\, \varphi(v), \quad t$ a closed term of L(C).

These latter are called the *quantifier axioms*. Their informal content is clear.

The set T_{Henkin} is not true in every L(C)-structure, but the next lemma shows that every L-structure can be turned into an L(C)-structure which is a model of T_{Henkin}, using the idea discussed following 4.5.

If $\mathrm{L} \subseteq \mathrm{L}'$ are languages and $\mathfrak{M}' = \langle M, F' \rangle$ is a structure for L′ then $\mathfrak{M} = \langle M, F' \upharpoonright \mathrm{L} \rangle$ is called the *reduct* of L′ to L and $\mathfrak{M}'$ is called an *expansion* of $\mathfrak{M}$ to the language L′. Thus, $\mathfrak{M}$ and $\mathfrak{M}'$ are the same except that $\mathfrak{M}'$ assigns meanings to the symbols in $\mathrm{L}' - \mathrm{L}$.

4.7. Lemma. *Let $\mathfrak{M}$ be any structure for a language* L *and let* L(C) *be the witnessing expansion of* L. *There is an assignment of elements of $\mathfrak{M}$ to the constant symbols of C so that the resulting expansion of $\mathfrak{M}$ is a model of T_{Henkin}.*

Proof. The quantifier axioms III, IV are going to be true regardless, so we only need worry about those of types I, II. Suppose we contrive to make those of type I true, and consider a typical one of type II:

$$\varphi(c_{\neg\varphi(v)}) \rightarrow \forall v\, \varphi(v).$$

Suppose the hypothesis is true in the expansion, but not the conclusion.

But then $\exists v\, \neg\varphi(v)$ is true, so by I, $\neg\varphi(c_{\neg\varphi(v)})$ is true, a contradiction. Thus, if all axioms of type I are true, so are all of type II.

We proceed to assign elements to the constants in C_n by induction on n. If $c_{\varphi(v)} \in C_1$ then $\exists v\, \varphi(v)$ is a sentence of L and hence makes sense in $\mathfrak{M}$. I $\mathfrak{M} \models \exists v\, \varphi(v)$, choose some $a \in M$ so that $\mathfrak{M} \models \varphi(a)$ and set $c^{\mathfrak{M}}_{\varphi(v)} = a$. I $\mathfrak{M} \models \neg\exists v\, \varphi(v)$, define $c^{\mathfrak{M}}_{\varphi(v)}$ arbitrarily. This makes all the positive Henkir axioms about the $c_{\varphi(v)} \in C_1$ true. But once the constants of C_1 are interpreted, all the sentences of $\mathrm{L}_1 = \mathrm{L} \cup C_1$ make sense, so we can carry out the same argument and assign elements to the $c_{\varphi(v)} \in C_2$, and so on. □

A *canonical structure* for L(C) is a structue $\mathfrak{M} = \langle M, \dots \rangle$ such that every $a \in M$ is denoted by some $c \in C$. That is, $M = \{c^{\mathfrak{M}} \mid c \in C\}$. The set Eq mentioned in 4.8 is the set of equality axioms of L(C) which are sentences of L(C); i.e., those which contain no variables.

The following lemma may seem rather technical but the equivalence o (i) and (iii) show that we have reduced problems about models of first-orde theories to essentially trivial questions about propositional logic. There is a price to be paid, however. Even in the case where the T in (i) is finite, th propositional theory in (iii) is infinite.

4.8. Main Lemma (The reduction to propositional logic). *Let* L *be a first-order language and let* L(C) *be the witnessing expansion of* L. *For an set T of sentences of* L *the following conditions are equivalent*:

(i) *T has a model; i.e. there is an* L-*structure $\mathfrak{M}$ which is a model of a sentences in T.*

(ii) *There is a canonical* L(C)-*structure $\mathfrak{M}$ which is a model of a sentences in T.*

(iii) *$T \cup T_{\text{Henkin}} \cup \mathrm{Eq}$ is consistent in the sense of propositional logic.*

Proof. The implication (ii) ⇒ (i) is immediate, while (i) ⇒ (iii) follow from Lemma 4.3. We prove (iii) ⇒ (ii). Let ν be a truth assignment to th prime sentences of L(C) such that $\nu(\varphi) = \mathbf{t}$ for all $\varphi \in T \cup T_{\text{Henkin}} \cup \mathrm{Eq}$. T prove the lemma, we construct a canonical model $\mathfrak{M} = \langle M, \dots \rangle$ such tha for all sentences φ of L(C),

$$\mathfrak{M} \models \varphi \quad \text{iff} \quad \bar{\nu}(\varphi) = \mathbf{t}.$$

The main function of T_{Henkin} is to guarantee that $\bar{\nu}$ satisfies the followir conditions:

$$\bar{\nu}(\exists v\, \varphi(v)) = \mathbf{t} \quad \text{iff} \quad \bar{\nu}(\varphi(c_{\varphi(v)})) = \mathbf{t},$$

$$\bar{\nu}(\forall v\, \varphi(v)) = \mathbf{t} \quad \text{iff} \quad \bar{\nu}(\varphi(t)) = \mathbf{t} \quad \text{for all closed terms } t \text{ of } \mathrm{L}(C).$$

The conditions allow us to construct our model $\mathfrak{M}$ out of the constants in C in a way that is analogous to the construction of a group from generators (the elements of C) and defining relations (the axioms of T). To define $\mathfrak{M}$ we must (a) specify the universe M of $\mathfrak{M}$, (b) define, for each n-ary relation symbol $R \in \mathrm{L}$ an n-ary relation $R^{\mathfrak{M}}$ to interpret R, (c) define for each n-ary function symbol $f \in \mathrm{L}$ an interpretation $f^{\mathfrak{M}}: M^n \to M$, and (d) define for each constant symbol c of $\mathrm{L} \cup C$ an element $c^{\mathfrak{M}} \in M$. Having thus constructed $\mathfrak{M}$ it will remain only to verify that $\mathfrak{M} \models \varphi$ iff $\bar{\nu}(\varphi) = \mathbf{t}$, for all sentences φ of $\mathrm{L}(C)$. This condition tells us how we must fulfill conditions (b)–(d) above.

(a) Define an equivalence relation $\approx$ on C by

$$c \approx d \quad \text{iff} \quad \nu((c = d)) = \mathbf{t}.$$

The equality axioms gurantee that $\approx$ is an equivalence relation on C. Suppose, for example, that $c \approx d$ and $d \approx e$. We check to see that $c \approx e$. Since $\nu(c = d) = \mathbf{t}$, $\nu(d = e) = \mathbf{t}$ by $c \approx d$, $d \approx e$ and since $\bar{\nu}(((c = d \wedge d = e) \to c = e)) = \mathbf{t}$ since this sentence is an equality axiom, $\nu(c = e) = \mathbf{t}$ so $c \approx e$. Let $\bar{c}$ be the equivalence class of c and let $M = \{\bar{c} \mid c \in C\}$.

(b) Define $R^{\mathfrak{M}}$ by

$$\langle \bar{c}_1, \ldots, \bar{c}_n \rangle \in R^{\mathfrak{M}} \quad \text{iff} \quad \nu(R(c_1, \ldots, c_n)) = \mathbf{t}.$$

To see that this is well defined we must check that if

$$\bar{c}_1 = \bar{d}_1, \ldots, \bar{c}_n = \bar{d}_n \quad \text{and} \quad \langle \bar{c}_1, \ldots, \bar{c}_n \rangle \in R^{\mathfrak{M}}$$

then $\langle \bar{d}_1, \ldots, \bar{d}_n \rangle \in R^{\mathfrak{M}}$. This is a consequence of the fact that

$$c_1 = d_1 \wedge \cdots \wedge c_n = d_n \wedge R(c_1, \ldots, c_n) \to R(d_1, \ldots, d_n)$$

is an equality axiom and hence is assigned true by $\bar{\nu}$.

(c) Let $c_1, \ldots, c_n \in C$ and $f \in \mathrm{L}$ be given. We claim there is a $c \in C$ such that $\nu(f(c_1 \cdots c_n) = c) = \mathbf{t}$. For consider the formula $\varphi(x)$ given by $(f(c_1 \cdots c_n) = x)$. If $\bar{\nu}(\exists v \varphi(v)) = \mathbf{t}$, then $\nu(f(c_1 \cdots c_n) = c_\varphi) = \mathbf{t}$. So suppose that $\nu(\exists v \varphi(v)) = \mathbf{f}$. But one member of T_{Henkin} is the sentence $(\varphi(f(c_1 \cdots c_n)) \to \exists v \varphi(v))$ so that $\bar{\nu}(\varphi(f(c_1 \cdots c_n)) = \mathbf{f}$. But this says that ν assigns $\mathbf{f}$ to the atomic sentence $(f(c_1 \cdots c_n) = f(c_1 \cdots c_n))$. But $\bar{\nu}(c_i = c_i) = \mathbf{t}$, $(i = 1, \ldots, n)$ and $\bar{\nu}((c_1 = c_1 \wedge \cdots \wedge c_n = c_n) \to (f(c_1 \cdots c_n) = f(c_1 \cdots c_n)) = \mathbf{t}$ since these are equality axioms, which is a contradiction. Thus $\bar{\nu}(\exists v \varphi(v)) = \mathbf{t}$ after all. We can define $f^{\mathfrak{M}}(\bar{c}_1, \ldots, \bar{c}_n) = \bar{c}$ for that $c \in C$ such that $\nu(f(c_1 \cdots c_n) = c) = \mathbf{t}$. An argument like that used in (b) shows that $f^{\mathfrak{M}}$ is well defined.

(d) If $c \in C$ let $c^{\mathfrak{M}} = \bar{c}$. If $d \in \mathrm{L}$, then an argument similar to that u
(c) shows that there is a $c \in C$ such that $\nu(d = c) = \mathbf{t}$ so let $d^{\mathfrak{M}} =$
this c.

This completes the construction of $\mathfrak{M}$ and guarantees that for a
sentences $\mathfrak{M} \models \varphi$ iff $\nu(\varphi) = \mathbf{t}$. To prove this for other sentences we pr
by induction on length of formulas. The propositional connective
trivial. For example, $\mathfrak{M} \models (\varphi \wedge \psi)$ iff $\mathfrak{M} \models \varphi$ and $\mathfrak{M} \models \psi$ (by definition
iff $\bar{\nu}(\varphi) = \bar{\nu}(\psi) = \mathbf{t}$ (by the induction hypothesis) iff $\bar{\nu}(\varphi \wedge \psi) = \mathbf{t}$. Supp
is $\exists x \psi(x)$. If $\bar{\nu}(\varphi) = \mathbf{t}$ then, by the condition above, there is a c suc
$\bar{\nu}(\varphi(c)) = \mathbf{t}$ so, by induction hypothesis, $\mathfrak{M} \models \psi(c)$ so $\mathfrak{M} \models \exists x \psi(x)$ so
On the other hand, if $\bar{\nu}(\varphi) = \mathbf{f}$ then $\bar{\nu}(\exists x \psi(x)) = \mathbf{f}$ so by T_{Henkin}, $\bar{\nu}(\psi($
for all closed terms t of $\mathrm{L}(C)$. In particular, for every $c \in C$, $\bar{\nu}(\psi(c$
By the induction hypothesis, $\mathfrak{M} \models \neg\psi(c)$ for all $c \in C$. Since every el
of M is denoted by some $c \in C$, $\mathfrak{M} \models \neg \exists x \psi(x)$. Thus $\mathfrak{M} \models \exists x \psi$
$\nu(\exists x \psi(x)) = \mathbf{t}$. The proof in the case when φ begins with $\forall$ is similar.

The Main Lemma provides a method for actually constructing mo
theories out of symbols. In particular, it gives us immediate proofs
Compactness and Lowenheim–Skolem Theorems.

Proof of the Compactness Theorem (2.4). Let T be a set of senter
the first-order language L such that every finite subset of T has a
We need to show that T has a model. By (iii) $\Rightarrow$ (i) of the Main Lemr
amounts to proving that $T \cup T_{\text{Henkin}} \cup \mathrm{Eq}$ is consistent in the se
propositional logic. But, by the Compactness Theorem for Propos
Logic, it suffices to prove that for every finite subset
$T_0 \cup T_{\text{Henkin}} \cup \mathrm{Eq}$ is consistent, which follows from the hypothes
(i) $\Rightarrow$ (iii) of the Main Lemma. □

Other proofs of this theorem appear in Chapters A.2 and A.3.
follows directly from the Completeness Theorem below.

Proof of the Löwenheim–Skolem Theorem (2.5). Let κ be some
cardinal and let L be a language with $\leq \kappa$ symbols. Since every forr
L is a finite sequence of symbols, there are $\leq \kappa$ formulas of L. Re
definition of the witnessing expansion $\mathrm{L}(C)$ of L, where $C =$
Clearly, by induction, each C_n has cardinality $\leq \kappa$ so C has car
$\leq \kappa$. Thus, any canonical structure for $\mathrm{L}(C)$ has $\leq \kappa$ elements,
desired result is an immediate consequence of (i) $\Rightarrow$ (ii) in the
Lemma. □

The Completeness Theorem for a Hilbert-style formal system **H**

There are several quite distinct approaches to the Completeness Theorem, corresponding to different ways of thinking about proofs. Within each of the approaches there are endless variations in the exact formulation, corresponding to which laws of thought are taken as basic, which as derived. We will ignore the minor variations. The different basic approaches are important, though, for different notions of proof lend themselves to different applications. The most important thing to remember, however, is that while there are many notions of *proof*, there is only one real notion of *provable* for first-order logic, as the Completeness Theorem shows.

The first type of formal system we discuss is a so called Hilbert-style formal system. It is usually the favorite of the mathematician because it is elegant and easy to remember. In 4.6 we sketch a Gentzen-type formal system. This type has proven very useful in analyzing the proof-theoretic strength of various mathematical theories. In a classroom situation, however, where students actually seem to enjoy working out formal proofs, the Fitch-type subordinate proof method or the Beth-type semantic tableaus are even better. The latter are discussed at length in SMULLYAN [1968].

In a Hilbert-style formal system, the emphasis is on logical *axioms*, keeping the rules of inference at a minimum. If we had taken $\exists x$ as a defined symbol, treating $\exists x$ as $\neg\forall x\neg\varphi$, the system would have been superficially even simpler. It seems somehow more to the point, however, to treat the laws of thought behind these quantifiers separately.

Let L be a fixed first-order language. All formulas below are first-order formulas of L, and all terms t are terms of L. Recall our convention in Section 3 about writing $\varphi(t/v)$, the result of replacing v by t in φ, only in case the variables in t do not occur bound in φ. We write $\varphi(t)$ for $\varphi(t/v)$ below.

Axiom Schemata of **H**

(1) All tautologies,

(2) All equality axioms,

(3) All formulas of either of the forms

$$(\forall v\varphi(v))\to\varphi(t), \qquad \varphi(t)\to\exists v\varphi(v).$$

Rules of Inference of **H**

(1) (*Modus Ponens*) From $(\varphi\to\psi)$ and φ infer ψ,

(2) (*Generalization rules*) If the variable v is not free in φ, then:

$$\text{from } \varphi\to\psi(v) \text{ infer } \varphi\to\forall y\,\psi(y),$$

$$\text{from } \psi(v)\to\varphi \text{ infer } \exists y\,\psi(y)\to\varphi.$$

These rules are usually written schematically as

$$\frac{(\varphi\to\psi)\quad\varphi}{\psi}$$

If v not free in φ, then:

$$\frac{\varphi\to\psi(v)}{\varphi\to\forall y\,\psi(y)};\qquad \frac{\psi(v)\to\varphi}{\exists y\,\psi(y)\to\varphi}$$

A *proof of* φ *from a set of sentences* T (in the formal system **H**) is a fini[te] sequence $\psi_1,\ldots,\psi_n$ of formulas, with $\psi_n=\varphi$, each of which is either a[n] axiom of **H**, a member of T, or else follows from earlier ψ_i by one of th[e] three rules of inference. We say that φ *is provable from* T, and write $T\vdash\varphi$, [if] there is a proof of φ from T.

4.9. GÖDEL COMPLETENESS THEOREM FOR **H**. *Let T be a set of sentences of [a] language* L. *A sentence* φ *is provable from T if and only if* φ *is true in a[ll] set-theoretic structures which are models of T. In symbols,* $T\models\varphi$ *iff* $T\vdash$

The easy half of the Completeness Theorem follows from the Soundne[ss] Lemma.

4.10. SOUNDNESS LEMMA. *Let* t *be a set of sentences,* $\mathfrak{M}$ *a model of T.* [If] $\varphi(v_1,\ldots,v_n)$ *is provable from T, then* $\mathfrak{M}\models\forall v_1\cdots\forall v_n\varphi(v_1,\ldots,v_n)$.

PROOF. One proves, by induction on n, that if $\psi_1,\ldots,\psi_n$ is a proof from [T,] then $\mathfrak{M}\models\forall v_1\cdots v_k\psi_n(v_1\cdots v_k)$. □

In the proof of the Completeness Theorem we need the followi[ng] lemma.

4.11. LEMMA. *Let T be a set of sentences.*

(i) *If* $T\vdash(\varphi\to\psi)$ *and* $T\vdash(\neg\varphi\to\psi)$, *then* $T\vdash\psi$.

(ii) *If* $T\vdash(\varphi\to\theta)\to\psi$, *then* $T\vdash(\neg\varphi\to\psi)$ *and* $T\vdash(\theta\to\psi)$.

(iii) *If* v *does not appear in* ψ *and if* $T\vdash[(\exists y\varphi(y)\to\varphi(v))\to\psi]$, *th[en]* $T\vdash\psi$.

(iv) *If v does not appear in* ψ *and if* $T\vdash(\varphi(v)\to\forall y\varphi(y))\to\psi$, *then* $T\vdash$

PROOF. (i) Notice that $[(\varphi \to \psi) \to ((\neg\varphi \to \psi) \to \psi)]$ is a tautology. Thus, if we write a proof of $(\varphi \to \psi)$ and apply modus ponens we get a proof of $(\neg\varphi \to \psi) \to \psi$. Then write a proof of $(\neg\varphi \to \psi)$ and apply modus ponens to get a proof of ψ.

(ii) Note, $[((\varphi \to \theta) \to \psi) \to (\neg\varphi \to \psi)]$ and $[((\varphi \to \theta) \to \psi) \to (\theta \to \psi)]$ are tautologies.

(iii) Suppose $T \vdash [(\exists y\, \varphi(y) \to \varphi(v)) \to \psi]$, where v is not free in ψ. By (ii), $T \vdash (\neg\exists y\, \varphi(y) \to \neg\psi)$ and $T \vdash \varphi(v) \to \psi$. Apply the second generalization rule and we have $T \vdash (\exists y\, \varphi(y) \to \psi)$. But then by (i), $T \vdash \psi$. The proof of (iv) is similar, but uses the first generalization rule. □

PROOF OF THE COMPLETENESS THEOREM. Suppose that $T \models \varphi$. By the Main Lemma (4.8) and the Compactness Theorem for propositional logic, there is a finite set $S \subseteq T \cup T_{\text{Henkin}} \cup \text{Eq}$ such that $S \cup \{\neg\varphi\}$ is inconsistent in the sense of propositional calculus. List the members of S in a list $\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_M$ as follows. The sequence $\alpha_1, \ldots, \alpha_N$ consists of those members of S which are either in $T \cup \text{Eq}$ or else are quantifier axioms (types III and IV) listed in any order. The β's are the members of S which are Henkin axioms of types I, II, but we must list them more carefully. Recall the languages $\text{L} = \text{L}_0 \subset \text{L}_1 \subset \cdots$ such that $\text{L}(C) = \bigcup_n \text{L}_n$. Define the rank of $\varphi \in \text{L}(C)$ to be the least n such that $\varphi \in \text{L}_n$. Now, choose for β_1 a Henkin axiom in S of maximum rank. Choose for β_2 a Henkin axiom in $S - \{\beta_1\}$ of maximum rank, etc. The point of arranging things in this way is that the witnessing constant about which β_i speaks, is not mentioned in $\beta_{i+1}, \ldots, \beta_M$. For example, if β_1 is

$$\exists v\, \eta(v) \to \eta(c_{\eta(v)}),$$

then $c_{\eta(v)}$ does not appear in any of the other $\beta_2, \ldots, \beta_M$, by the maximality condition on β_1.

Recalling that $S \cup \{\neg\varphi\}$ is not consistent in the sense of propositional logic, and associating parentheses to the right, we see that

$$(\alpha_1 \to \alpha_2 \to \cdots \to \alpha_N \to \beta_1 \to \cdots \to \beta_M \to \varphi)$$

is a tautology. Replace each witnessing constant in this sentence by a distinct new variable. The result is still a tautology:

$$\alpha'_1 \to \alpha'_2 \to \cdots \to \alpha'_N \to \beta'_1 \to \cdots \to \beta'_M \to \varphi'$$

but $\varphi' = \varphi$ since φ has no witnessing constants in it. Each $\alpha'_1, \ldots, \alpha'_N$ is either a logical axiom or else is a member of T, so we may apply modus ponens N times and obtain a proof of:

$$\beta'_1 \to \cdots \to \beta'_M \to \varphi.$$

But now we apply parts (iii) and (iv) of Lemma 4.11 to succesively remove $\beta'_1, \beta'_2, \ldots, \beta'_M$ and obtain a proof of φ. □

Notice that our proof used only the derived rules 4.11 (iii), (iv) so we could have used them in place of the more standard rules of **H**. This is discussed in SMULLYAN [1968].

The Completeness Theorem for a Gentzen-type formal system $\mathbf{G}^+$

Hilbert-style systems are easy to define and admit a simple proof of the Completeness Theorem but they are difficult to use. Gentzen systems reverse this situation by emphasizing the importance of inference rules, reducing the role of logical axioms to an absolute minimum.

We use Γ, Δ to range over finite sets of formulas. A *sequent* is a pair $\langle \Gamma, \Delta \rangle$ which is written $\Gamma \vdash \Delta$ and read, informally, as Γ yields Δ or, rather the *conjunction* of all the formulas in Γ yields the *disjunction* of all the formulas in Δ. We write Γ, φ for $\Gamma \cup \{\varphi\}$.

We first restrict attention to propositional logic.

Axioms: $\Gamma, \varphi \vdash \Delta, \varphi$.

Rules:

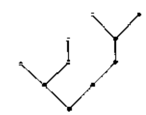
$$(\wedge\vdash)\quad \frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, (\varphi \wedge \psi) \vdash \Delta} \qquad (\vdash\wedge)\quad \frac{\Gamma \vdash \Delta, \varphi \qquad \Gamma \vdash \Delta, \psi}{\Gamma \vdash \Delta, (\varphi \wedge \psi)}$$

$$(\vee\vdash)\quad \frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma, \psi \vdash \Delta}{\Gamma, (\varphi \vee \psi) \vdash \Delta} \qquad (\vdash\vee)\quad \frac{\Gamma \vdash \Delta, \varphi, \psi}{\Gamma \vdash \Delta, (\varphi \vee \psi)}$$

$$(\neg\vdash)\quad \frac{\Gamma \vdash \Delta, \varphi}{\Gamma, \neg\varphi \vdash \Delta} \qquad (\vdash\neg)\quad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta, \neg\varphi}$$

$$(\to\vdash)\quad \frac{\Gamma \vdash \Delta, \varphi \qquad \Gamma, \psi \vdash \Delta}{\Gamma, (\varphi \to \psi) \vdash \Delta} \qquad (\vdash\to)\quad \frac{\Gamma, \varphi \vdash \Delta, \psi}{\Gamma \vdash \Delta, (\varphi \to \psi)}$$

A *derivation* in this system is a finite tree of formulas like, e.g.



with axioms at the upper nodes and such that each sequent on the tree follows from the ones immediately above it by one of the rules. Rather than define this precisely, we given an example, a derivation of the sequent $((\varphi \wedge \neg\psi) \vee \theta) \vdash (\varphi \vee \theta)$:

$$\dfrac{\dfrac{\dfrac{\varphi, \neg\psi \vdash \varphi, \theta \quad \text{(axiom)}}{(\varphi \wedge \neg\psi) \vdash \varphi, \theta} \text{ (by } \wedge\vdash) }{(\varphi \wedge \neg\psi) \vdash (\varphi \vee \theta)} \text{ (by } \vdash\vee) \qquad \dfrac{\theta \vdash \varphi, \theta \quad \text{(axiom)}}{\theta \vdash (\varphi \vee \theta)} \text{ (by } \vdash\vee)}{((\varphi \wedge \neg\psi) \vee \theta) \vdash (\varphi \vee \theta)} \text{ (by } \vee\vdash)$$

4.12. THE COMPLETENESS THEOREM FOR THE PROPOSITIONAL FRAGMENT OF G. *Let Γ, Δ be finite sets of propositional formulas. The following are equivalent*:

(i) *Every truth assignment ν making all $\varphi \in \Gamma$ true makes at least one $\psi \in \Delta$ true.*

(ii) *There is a derivation of $\Gamma \vdash \Delta$ using the above axioms and rules.*

PROOF. The proof of (ii) $\Rightarrow$ (i) is easy by induction on the length of the derivation. For the proof of (i) $\Rightarrow$ (ii), start with a pair (Γ, Δ) satisfying (i). We attempt to build a derivation of $\Gamma \vdash \Delta$ by working backwards. At each step, we work on a formula in $\Gamma \cup \Delta$ of maximal length, breaking it apart by means of one of the rules. For example, if $(\psi \to \theta) \in \Gamma$ is the longest formula in $\Gamma \cup \Delta$ then, at the first stage, we write down

$$\frac{\Gamma_0 \vdash \Delta, \psi \qquad \Gamma_0, \varphi \vdash \Delta}{\Gamma_0, (\psi \to \theta) \vdash \Delta} \quad (\to\vdash)$$

where $\Gamma_0 = \Gamma - \{(\psi \to \theta)\}$. We now work on $\Gamma_0 \vdash \Delta, \psi$ and $\Gamma_0, \varphi \vdash \Delta$ separately. Eventually we end up with sequents which cannot be broken down further. If each of the sequents on the ends is an axiom, i.e., has a prime formula common to both sides, then we have constructed a derivation of $\Gamma \vdash \Delta$. So suppose that one of these end nodes $\Gamma' \vdash \Delta'$ is not an axiom. Define ν on $\Gamma' \cup \Delta'$ by

$$\nu(p) = \begin{cases} \mathbf{t} & \text{if } p \in \Gamma', \\ \mathbf{f} & \text{if } p \in \Delta' \end{cases}$$

and define ν arbitrarily on other prime formulas. This is possible since $\Gamma' \cap \Delta' = \emptyset$. A case by case examination of the rules shows that every sequent $\Gamma'' \vdash \Delta''$ beneath $\Gamma' \vdash \Delta'$ also gets $\mathbf{t}$ assigned to everything on the left by $\bar{\nu}$ but $\mathbf{f}$ to everything on the right. In particular, this happens to $\Gamma \vdash \Delta$, a contradiction. □

To pass from propositional logic to first-order logic we add equality axioms, an equality rule and four quantifier rules.

Equality axioms: $\Gamma \vdash \Delta, (t = t)$ (t any term).

Equality rule: If E is $(t_1 = t_2)$ or $(t_2 = t_1)$, then

$$\frac{\Gamma, \varphi(t_1) \vdash \Delta, \psi(t_1)}{\Gamma, E, \varphi(t_2) \vdash \Delta, \psi(t_2)};$$

$$(\forall\vdash) \quad \frac{\Gamma, \varphi(t) \vdash \Delta}{\Gamma, \forall v \varphi(v) \vdash \Delta}. \qquad (\vdash\exists) \quad \frac{\Gamma \vdash \Delta, \varphi(t)}{\Gamma \vdash \Delta, \exists v \varphi(v)}$$

In the next two rules, *the variable v is not allowed to occur free in $\Gamma \cup \Delta$*:

$$(\vdash\forall) \quad \frac{\Gamma \vdash \Delta, \varphi(v)}{\Gamma \vdash \Delta, \forall y \varphi(y)}. \qquad (\exists\vdash) \quad \frac{\Gamma, \varphi(v) \vdash \Delta}{\Gamma, \exists y \varphi(y) \vdash \Delta}$$

The formal system **G** has the above axioms and rules of inference. The system $\mathbf{G}^+$ has, in addition, a rule called "cut" which is the counterpart of modus ponens:

$$\text{cut:} \quad \frac{\Gamma, \varphi \vdash \Delta \qquad \Gamma \vdash \Delta, \varphi}{\Gamma \vdash \Delta}$$

Given a set T of sentences of L, we say that a formula φ is derivable from T in $\mathbf{G}^+$ if there is a derivation of $\Gamma \vdash \{\varphi\}$ for some finite $\Gamma \subseteq T$.

4.13. THE COMPLETENESS THEOREM FOR $\mathbf{G}^+$. *A sentence φ is derivable from a set T of sentences in the system $\mathbf{G}^+$ iff $T \models \varphi$.*

PROOF. The ($\Rightarrow$) direction follows by a Soundness Lemma entirely analogous to 4.10. To prove ($\Leftarrow$) we use the Main Lemma, the Compactness Theorem for propositional logic and 4.12. By the Main Lemma and the Compactness Theorem for Propositional Logic, there is a finite $S \subseteq T \cup T_{\text{Henkin}} \cup \text{Eq}$ such that every truth valuation ν making S true makes φ true. Using the exact notation as in the proof of 4.9, let $S = \{\alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N\}$, and let $\alpha'_1, \ldots, \alpha'_N, \beta'_1, \ldots, \beta'_M$ be as before. Thus, if $\Gamma = \{\alpha'_1, \ldots, \alpha'_N, \beta'_1, \ldots, \beta'_M\}$, we see that $\Gamma \vdash \varphi$ is derivable in $\mathbf{G}^+$ using only the axioms and rules of propositional logic, by 4.12. Let $\Gamma_0 = \{\alpha'_1, \ldots, \alpha'_K\}$, $K \le N$, be the subset of $\{\alpha'_1 \cdots \alpha'_N\}$ of members of T. We can turn the derivation of $\Gamma \vdash \varphi$ into a derivation of $\Gamma_0 \vdash \varphi$ by means of the following claims.

(1) If α is an equality axiom or a quantifier axiom III or IV, then $\emptyset \vdash \varphi$.

(2) If $\Gamma \vdash \Delta$, then $\Gamma \cup \Gamma' \vdash \Delta \cup \Delta'$.

These are simple and from them we obtain $\Gamma_0 \vdash \alpha'_i$ for all $J < i \le N$ so we can apply cut $N - J$ times and get a derivation of $\Gamma_0 \cup \{\beta'_1, \ldots, \beta'_M\} \vdash \varphi$.

(3) If $\Gamma, (\varphi \to \theta) \vdash \Delta$, then $\Gamma \vdash \Delta, \varphi$ and $\Gamma, \theta \vdash \Delta$.

This is similar to 4.11(ii). We prove the first.

$$\dfrac{\dfrac{\dfrac{\vdots}{\Gamma, \varphi \to \theta \vdash \Delta}\ \text{(hypothesis)}}{\Gamma, \varphi \to \theta \vdash \Delta, \varphi}\ \text{(by 2)} \qquad \dfrac{\dfrac{}{\Gamma, \varphi \vdash \Delta, \theta, \varphi}\ \text{(axiom)}}{\Gamma \vdash \Delta, (\varphi \to \theta), \varphi}\ (\vdash\to)}{\Gamma \vdash \Delta, \varphi}\ \text{(cut)}$$

The second is similar. Combining these, the rules $(\vdash \forall)$ and $(\exists \vdash)$, and cut we obtain:

(4) If $\Gamma, \exists y\, \varphi(y) \to \varphi(v) \vdash \Delta$ and v is not free in $\Gamma \cup \Delta$, then $\Gamma \vdash \Delta$.

(5) If $\Gamma, \varphi(v) \to \forall v\, \varphi(v) \vdash \Delta$ and v not free in $\Gamma \cup \Delta$, then $\Gamma \vdash \Delta$.

Using these derived rules we remove $\beta'_1, \ldots, \beta'_M$ from the hypothesis and obtain a derivation of $\Gamma_0 \vdash \varphi$. □

We have used the cut rule heavily in our proof of the Completeness Theorem for $\mathbf{G}^+$, but it did not enter into the proof for the propositional part. Is it really necessary? No. One can, by working directly with the system **G**, and expanding on the proof of the propositional part, prove the completeness of the system **G**. See, e.g. SMULLYAN [1968]. This shows that the cut rule can be eliminated. Historically, things went the other way round, and constituted the first important chapter in proof theory after Gödel's Incompleteness Theorems.

4.14. CUT-ELIMINATION THEOREM (Gentzen). *Any sequent which has a derivation with the cut rule has one without it. I.e.,* $\mathbf{G}^+$ *and* **G** *have the same derivable sequents.*

Gentzen's proof was by a complicated double induction, showing how to transform any derivation allowing the cut rule into a derivation without cut. By analyzing such inductive proofs one is able to a obtain precise least upper bound to the methods of induction which are provable in first-order arithmetic. This topic is treated in detail in Chapter D.2 on cut-elimination. The system used there is simpler since only formulas in so called "negation normal form" are considered.

This concludes our discussion of the Completeness Theorem. We leave the reader with the instructive exercise of proving $\exists y\, [\varphi(y) \to \forall x\, \varphi(x)]$ in the systems **H**, **G** and $\mathbf{G}^+$. The informal proof is: pick y so that $\neg\varphi(y)$, if there is such a y; otherwise let y be arbitrary. Thus, if $\varphi(y)$, then $\forall x\, \varphi(x)$. The proof in $\mathbf{G}^+$ is similar.

5. Beyond first-order logic

Many logicians would contend that there is no logic beyond first-orde logic, in the sense that when one is forced to make all one's mathematica (extra-logical) assumptions explicit, these axioms can always be expresse in first-order logic, and that the informal notion of *provable* used i mathematics is made precise by the formal notion *provable in first-ord logic*. Following a sugestion of Martin Davis, we refer to this view a *Hilbert's Thesis.*

The first part of Hilbert's Thesis, that all of classical mathematics i ultimately expressible in first-order logic, is supported by empirical ev dence. It would indeed be revolutionary were someone able to introduce new notion which was obviously part of logic. The second part of Hilbert Thesis would seem to follow from the first part and Gödel's Completene Theorem. Thus Hilbert's Thesis is, to some extent, accepted by man mathematical logicians.

Even those who accept Hilbert's Thesis in theory, however, are a far c from accepting it in practice. It would be completely impractical and, i fact, counter-productive, to always make all one's extra-logical assum tions explicit.

Let us reconsider a couple of examples from Section 2.

Example. The axiom $\forall x\, \exists n \geq 1\, (nx = 0)$ expressing the torsion propert for abelian groups is not a first-order axiom (by 2.3). If we were to app Hilbert's Thesis in this case, we would have to axiomatize not only grou theory but also the properties of natural numbers needed to carry out th arguments we were after. This would mean that the theory of torsic groups encompasses all of first-order number theory, something clearly n in the spirit of modern algebra.

Example. The notions of metric space and Hilbert space are relative simple, modulo the ordered field $\mathbb{R}$ of real numbers. As we saw in Sectic 2, however, $\mathbb{R}$ is only categorical relative to set theory. It would be count productive, though, even to pretend to formulate all ones theorems abo metric spaces within some formal system of set theory, when most of wh one wants to do is first-order modulo the field $\mathbb{R}$. There is no point saddling the study of metric spaces with any more of the problems inhere in set theory, unsolvable problems the likes of which mathematics h never known before, than necessary.

The examples show that it is common mathematical practice to accept certain notions and structures as basic and work axiomatically from there on, even when we are aware that these notions cannot be completely axiomatized in the restricted language of first-order logic in and of themselves. The algebraist constantly takes the notion *finite* as basic. The student of analysis, metric spaces and Hilbert spaces begins with the structure $\mathbb{R}$ of reals. Logicians have developed strengthenings of first-order logic which allow him to be more faithful to this mathematical practice, logics which absorb certain mathematical notions, or structures, into the logic, in the same way that the algebraist attempts to absorb the notion of *finite* into his informal logic. In this section we briefly discuss some of these extensions of first-order logic.

5.1. Many-sorted first-order logic

Two-sorted first-order logic is just like ordinary first-order logic except that one has two distinct sorts of variables. For example, the natural way of writing the axioms for vector spaces is to have one sort of variable $r, s, t, \ldots$ over scalars (elements of a field $\mathfrak{F}$) and a different sort $v, w, \ldots$ over vectors. A vector space consists of a triple $(\mathfrak{F}, \mathfrak{V}, \cdot)$ where $\mathfrak{F}$ is a field, $\mathfrak{V} = \langle V, +, 0\rangle$ the structure of vectors with vector addition, and the operation $\cdot$ of scalar multiplication. In general, a two-sorted structure $(\mathfrak{M}, \mathfrak{N}, \ldots)$ consists of two ordinary structures plus some functions and relations on their union. Two-sorted (or many-sorted) logic is only superficially stronger, though often more natural, than ordinary logic, since we can always take a structure $(\mathfrak{M}, \mathfrak{N}, \ldots)$ and turn it into a ordinary structure $\langle M \cup N, M, N, \ldots, \ldots, \ldots\rangle$ with unary predicates M and N to sort out the different sorts of elements. This reduction allows most results of first-order logic to be transferred to many-sorted logic, and is part of the evidence for Hilbert's Thesis. Malcev and Feferman, among others, have stressed the advantages of working directly with the many-sorted case. A good introduction can be found in FEFERMAN [1974].

5.2. ω-logic

If we take a two-sorted language and consider two-sorted structures $(\mathfrak{M}, \mathfrak{N}, \ldots)$ with a *fixed structure* $\mathfrak{N}$, then we obtain so called $\mathfrak{N}$-logic. For example, $\mathbb{R}$-logic is appropriate to the study of metric spaces and real Hilbert spaces. For $\mathbb{N}$ the structure of natural numbers, $\mathbb{N}$-logic is usually called ω-logic. It is appropriate to the study of, say, Euclidean rings, since a Euclidean ring is a ring $\mathfrak{R}$ with a function $d : \mathfrak{R} \to \mathbb{N}$ satisfying the usual first-order axioms. As long as $\mathfrak{N}$ is infinite, $\mathfrak{N}$-logic is stronger than

first-order logic. For example, the Compactness Theorem must fail fo $\mathfrak{N}$-logic.

5.3. Weak second-order logic

Weak second-order logic is an attempt to build the notion of *finite* int logic in a natural way. Let L be a given first-order language. Let x, y, z b the variables of L. From L we form a new two-sorted language L* whic has variables a, b, c and a membership symbol $\in$. Given a structur $\mathfrak{M} = \langle M, \ldots\rangle$ for L we expand it to a structure $\mathrm{HF}(\mathfrak{M})$ for L*, called *th structure of hereditarily finite sets on* $\mathfrak{M}$, as follows. Let

$$\mathrm{HF}_0(M) = \emptyset$$

$$\mathrm{HF}_{n+1}(M) = \{\text{all finite subsets of } M \cup \mathrm{HF}_n(M)\},$$

$$\mathrm{HF}(M) = \bigcup_n \mathrm{HF}_n(M).$$

Then $\mathrm{HF}(\mathfrak{M}) = (\mathfrak{M}, \mathrm{HF}(M), \in \restriction (M \cup \mathrm{HF}(M)))$. In weak second-ord logic (more accurately called weak finite-type logic) we allow ourselves t use formulas of L* and to interpret the set variables a, b, c over HF(M Anyone familiar with the development of intuitive set theory in, say, Z will realize that we can define the natural numbers in $\mathrm{HF}(\mathfrak{M})$, and th notions of finite sequence in $\mathrm{HF}(\mathfrak{M})$. In fact, $\mathrm{HF}(\mathfrak{M})$ is admissible (se Chapter A.7 on admissible sets, in particular, 3.1 and 2.16), so we ca define functions by recursion. In particular, all of the sentences of Section which we called weak second-order are easily seen to be weak secon order in this precise sense. Weak second-order logic has essentially th same strength as ω-logic, but is much more natural in the context algebra, since one can work directly with integers, finite set, fini sequences, etc.

5.4. Infinitary logic

Weak second-order logic attempts to absorb the notion of finite into t semantics (meaning) of the logic. It has turned out to give a more elega theory, however, to absorb it into syntax of the logic by allowing infini formulas, like

$$\forall x\,[x = 0 \vee 2x = 0 \vee \cdots].$$

The logic $L_{\omega_1\omega}$ allows the additional formation rule: if Φ is a countable s of formulas then $\bigwedge \Phi$ (the conjunction of Φ) and $\bigvee \Phi$ (the disjunction Φ) are formulas. This logic is discussed in several of the chapters in t part of the book. The notation $L_{\omega_1\omega}$ is explained by the fact that countable ($< \omega_1$) conjunctions and disjunctions are permitted but only finite ($< \omega$) strings of quantifiers. One can think of the logic $L_{\omega_1\omega}$ as expressing those notions which are first-order modulo a countable amount of information. The Lowenheim–Skolem Theorem holds for $L_{\omega_1\omega}$, but not the Compactness Theorem. To get a Completeness Theorem, for countable theories T of $L_{\omega_1\omega}$, one must add an infinitary rule of proof.

It is easy to translate weak second-order logic into $L_{\omega_1\omega}$, but not vice-versa. In particular, in weak second-order logic there are relations which are implicitly, but not explicity definable, something that cannot happen in first-order logic or $L_{\omega_1\omega}$, by Beth's Theorem. If one looks for the *smallest* logic containing weak second-order logic in which all implicitly definable relations are explicitly definable, then one is led to the study of admissible fragments L_A of $L_{\omega_1\omega}$, as studied in Chapter A.7.

5.5. Logic with new quantifiers

It is easy to see that all finitary propositional *connectives* are definable from the ones in first order logic, in fact from $\vee$ and $\neg$. Mostowski long ago raised the possibility of adding new *quantifiers* to first-order logic. Thus, let Q be a new symbol and allow the formation rule: if $\varphi(x)$ is a formula, so is $Qx\,\varphi(x)$. There are many possible interpretations of Q. For example, we could define $\mathfrak{M} \models Qx\,\varphi(x)$ iff there are *infinitely many* x such that $\mathfrak{M} \models \varphi(x)$. This logic, called $L(Q_0)$, is essentially equivalent to ω-logic and weak second-order logic.

If we define $\mathfrak{M} \models Qx\,\varphi(x)$ iff there are *uncountable many* x such that $\mathfrak{M} \models \varphi(x)$ then we obtain logic with the quantifier "there exists uncountably many". This logic, unlike all the others mentioned earlier, has a Completeness Theorem and Compactness Theorem entirely analogous to that of ordinary first-order logic, (as long as the set L of symbols is finite or countable). Few people would claim that the notion of *uncountable* is a logical, rather than mathematical, notion, but the Completeness Theorem of KEISLER [1970] for this logic does give one pause. The notions of "many" and "most" seem almost logical and various precise mathematical notions like "measure 1", "second category", "infinitely many", and "uncountably many" use the intuitive notions for motivation. The Completeness Theorem of Keisler for "there exists uncountably many" shows that this notion provides a mathematicallly precise model for one informal concept of "many". Written out in English, using "many" and "few" for "uncountable" and "not uncountable" respectively, Keisler's basic axioms are:

(1) For all y there are few x such that $x = y$.

(2) If $\varphi(x) \to \psi(x)$ for all x, and if many x satisfy φ then many x satisfy ψ.

(3) If many x satisfy $(\varphi(x) \vee \psi(x))$, then either many x satisfy φ or many x satisfy ψ.

(4) If there are only a few x for which $\exists y\,\varphi(x, y)$, and if for each x there are only a few y such that $\varphi(x, y)$, then there are only a few y for which $\exists x\,\varphi(x, y)$.

Notice that "there are many x" is not a consequence of the axioms since the axioms hold in all structures, finite, countable, or uncountable. Sometimes, late at night, one can almost imagine some other world where such axioms are considered laws of thought in the same way that we accept the laws of first-order logic. But now we are entering the realm of science fiction, or mathematical fiction, so we had better stop.

5.6. Abstract model theory

Recent years have witnessed the foundation of a new branch of model theory. Abstract model theory steps back and surveys the whole spectrum of logics and the relationships between them. A *logic* consists of a *syntax* and a *semantics* which fit together nicely, in the sense that elementary syntactic operations (like renaming symbols) are performable and have their desired meaning. Glancing at the above examples should give one a feeling for a more precise definition (see BARWISE [1974]).

Of the above examples, weak second-order logic, $L_{\omega_1\omega}$ and $L(Q_0)$ all satisfy the Löwenheim–Skolem Theorem (Theorem 2.5 with $\kappa = \aleph_0$) but not the Compactness Theorem. On the other hand, $L(Q)$, where Q means uncountable, satisfies the Compactness Theorem but not the Löwenheim–Skolem Theorem. This is explained by one of the first, and still most striking, results of abstract model theory. A proof can be found in BARWISE [1974] where other references are also given.

5.7. THEOREM (Lindström). *First-order logic is the only logic closed under* $\wedge$, $\neg$, $\exists$ *which satisfies the Compactness Theorem and the Löwenheim–Skolem Theorem.*

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A.2

Fundamentals of Model Theory

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Contents

HANDBOOK OF MATHEMATICAL LOGIC
Edited by J. Barwise