Reductions and Causality (VI)

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Escuela de Ciencias Informáticas
Universidad de Buenos Aires

July 26, 2013

http://jeanjacqueslevy.net/courses/13eci

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Plan

- complete reductions
- sublattice of complete reductions
- more on canonical representatives
- costs of reductions + sharing
- speculative computations
- semantics with Böhm trees

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Labeled λ-calculus

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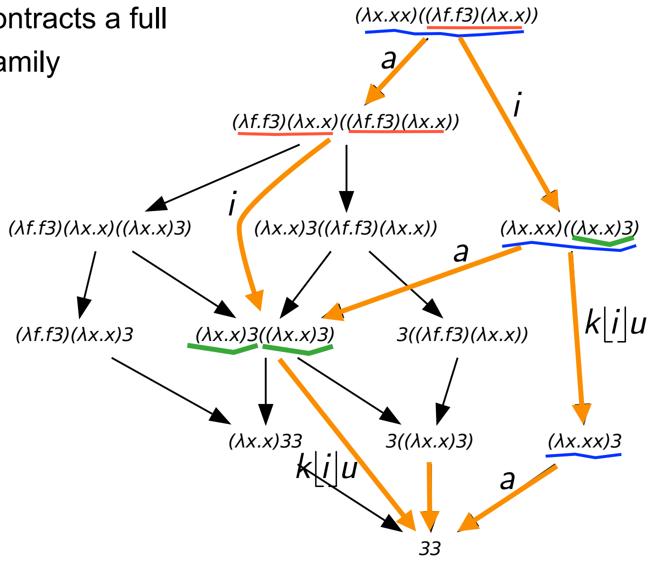
Complete reductions (1/5)

- Each step contracts a full single redex family



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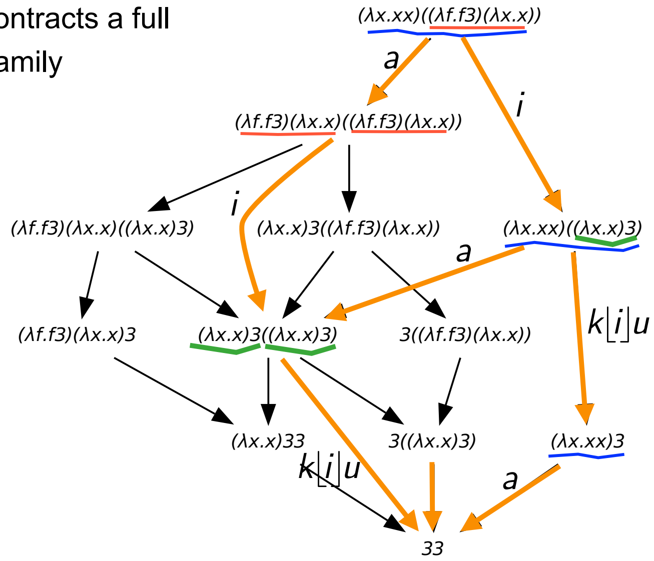

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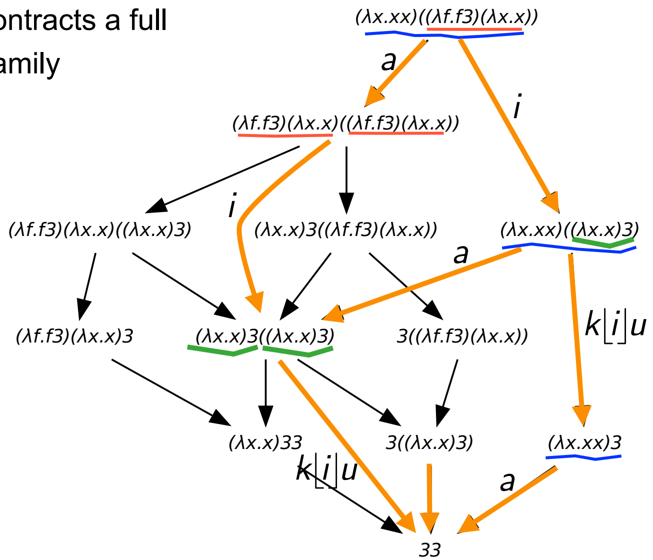

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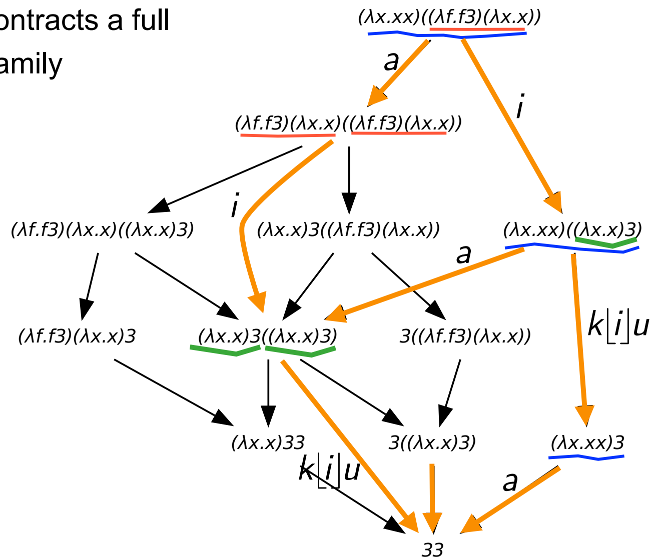

Complete reductions (2/5)

- **Definition** [complete reductions]

$\langle \rho, \mathcal{F} \rangle$ is an historical set of redexes when $\mathcal{F}$ is a set of redexes in final term of ρ.

$\langle \rho, \mathcal{F} \rangle$ is **f-complete** when it is maximum set such that

$$R, S \in \mathcal{F} \text{ implies } \langle \rho, R \rangle \sim \langle \rho, S \rangle$$

An **f-complete reduction** contracts an f-complete set at each step.

- **Proposition** [lattice of f-complete reductions]

Complete reductions form a sub-lattice of the lattice of reductions.

Proof simple use of following lemma which implies f-complete parallel moves.

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Complete reductions (3/5)

- **Notations**
  $M \overset{\alpha}{\Longrightarrow} N$ when $M \xrightarrow{\mathcal{F}} N$ and $\mathcal{F}$ is the set of redexes with name α in M.

  MaxRedNames(M) when all redexes in M have maximal names.

- **Lemma** `[complete reductions preserve max redex names]`
  $M \overset{\alpha}{\Longrightarrow} N$ and MaxRedNames(M) implies MaxRedNames(N)

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An **d-complete reduction** contracts a d-complete set at each step.

- **Proposition** [below canonical representative]
  Let $\langle \rho_0, R_0 \rangle$ be canonical representative in its family.
  Let $\rho_0 \sqsubseteq \rho$. Then $\langle \rho_0, R_0 \rangle \sim \langle \rho, R \rangle$ iff $\langle \rho_0, R_0 \rangle \leq \langle \rho, R \rangle$.

  Proof difficult.

- **Proposition** [f-complete = d-complete]
  d-complete reductions coincide with f-complete reductions.

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- **Proposition** [f-complete = d-complete]
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Complete reductions (5/5)

- **Proposition** [length of reduction = number of families]

  In complete reductions, number of steps equals the number of contracted redex families.

  Proof application of MaxRedNames lemma.

- **Corollary** [optimal reductions]

  In complete reductions, never redex of same family is contracted twice.

- **Implementation** [optimal reductions]

  Can we implement efficiently complete reductions ?

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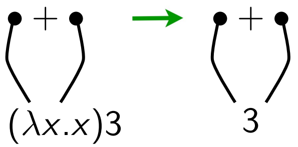
Implementation (1/5)

- **Implementation** [optimal reductions] algorithm [John Lamping, 90 -- Gonthier-Abadi-JJ, 91]

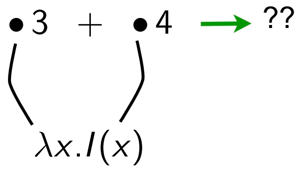
- Sharing of basic values is easy:

$(\lambda x.x + x)((\lambda x.x)3) \longrightarrow$



- Problem is sharing of functions:

$(\lambda x.x3 + x4)((\lambda x.I(x)) \longrightarrow$



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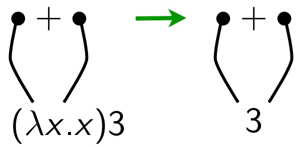

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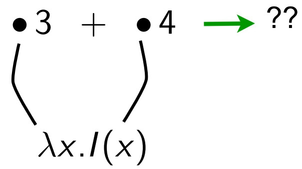

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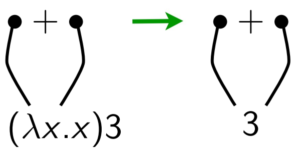

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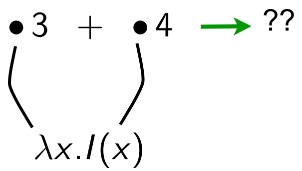

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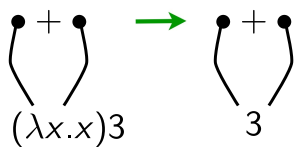

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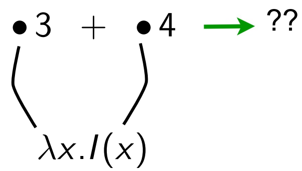

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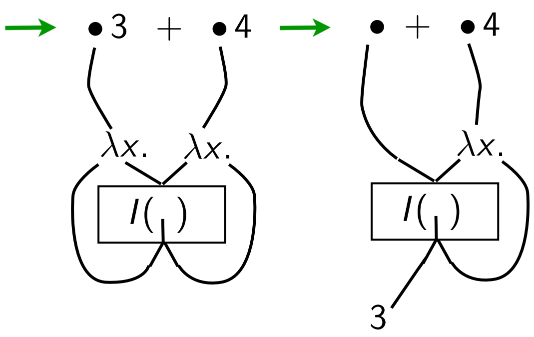

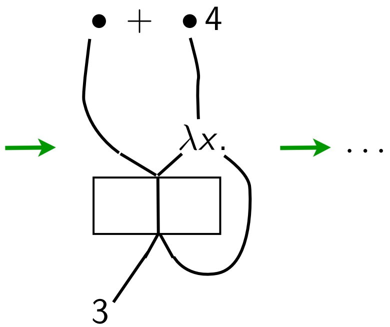

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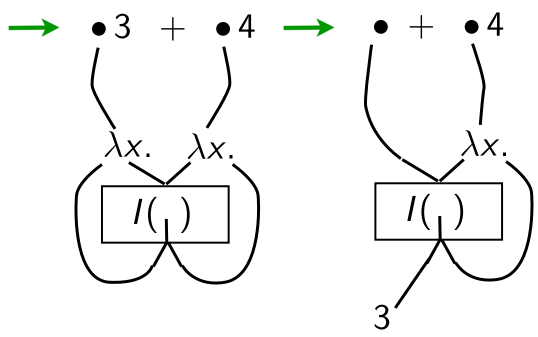

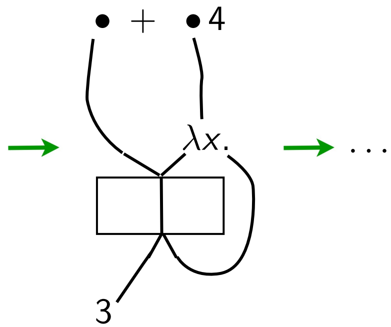

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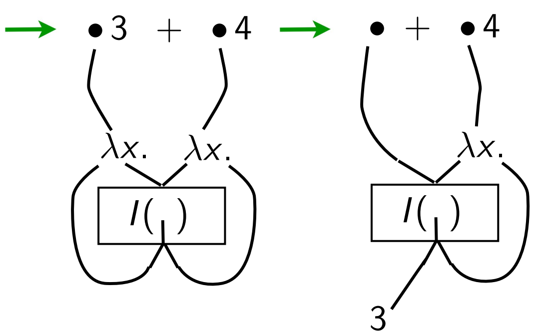

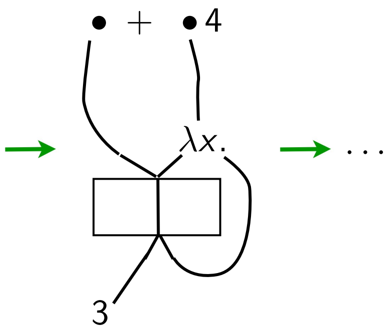

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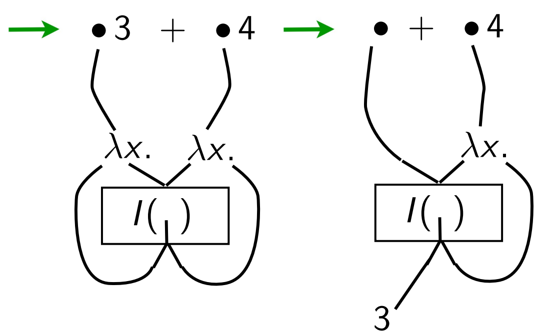

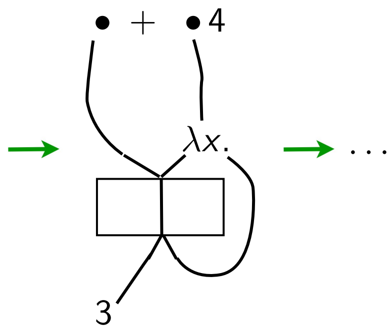

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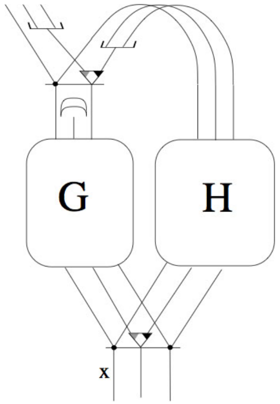

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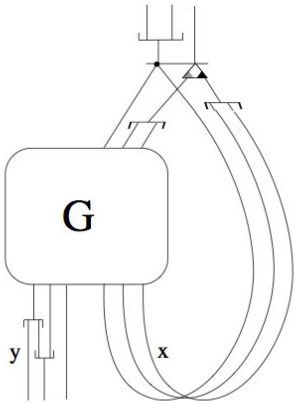

λ-abstraction

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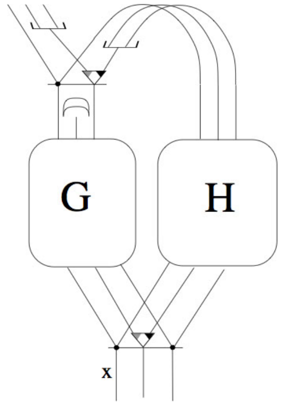

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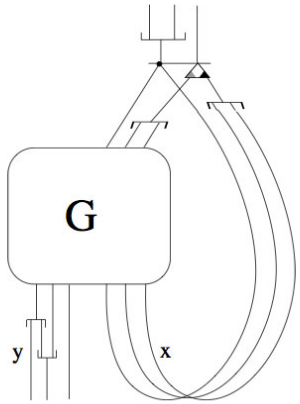

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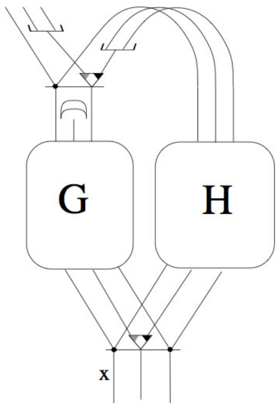

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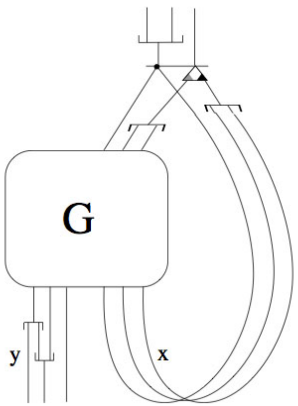

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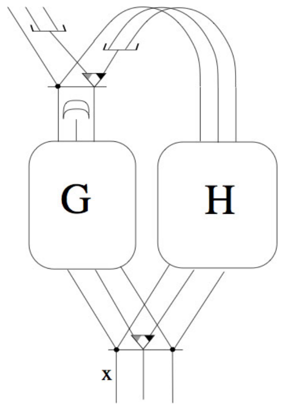

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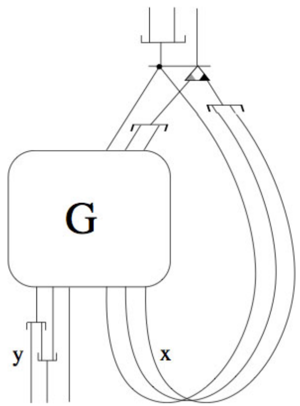

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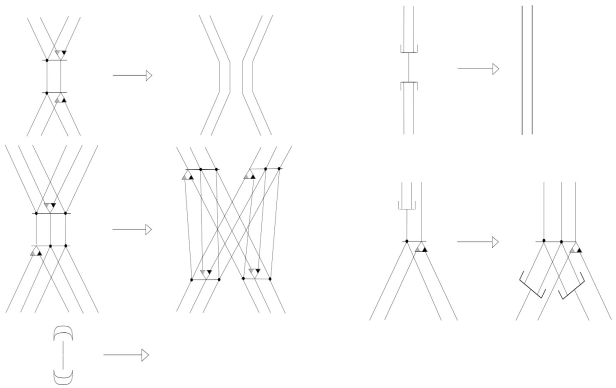
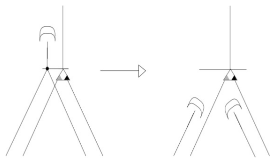
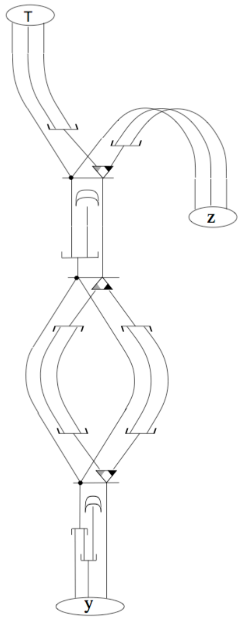


rules





$(\lambda x.yx)z$

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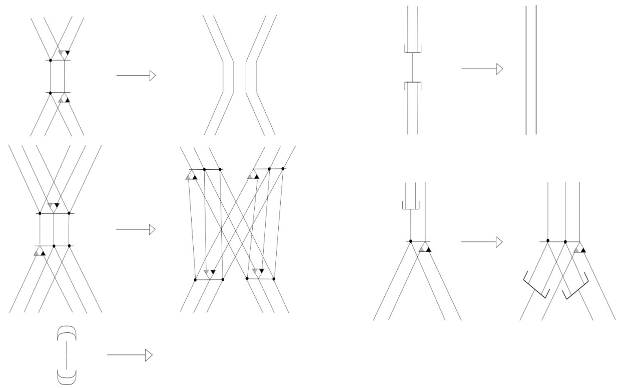

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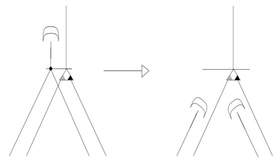

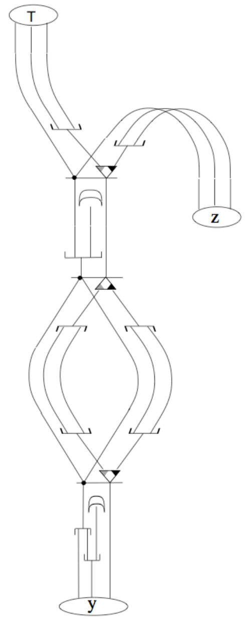

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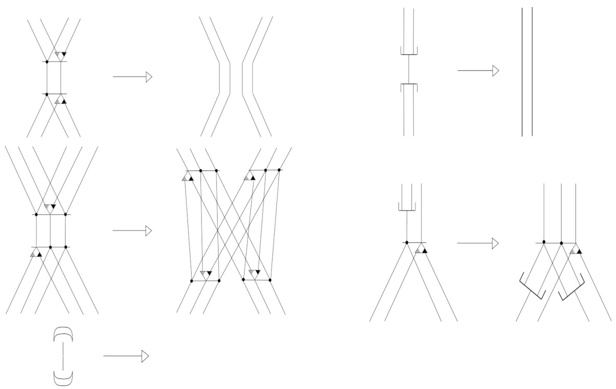

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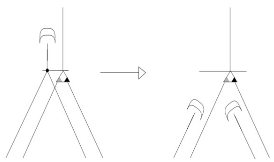

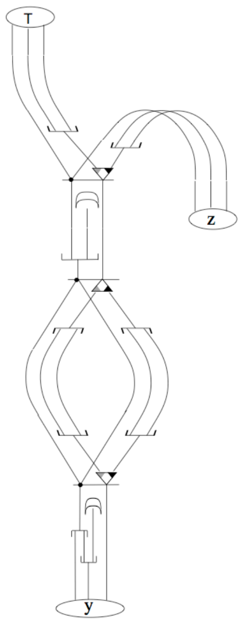

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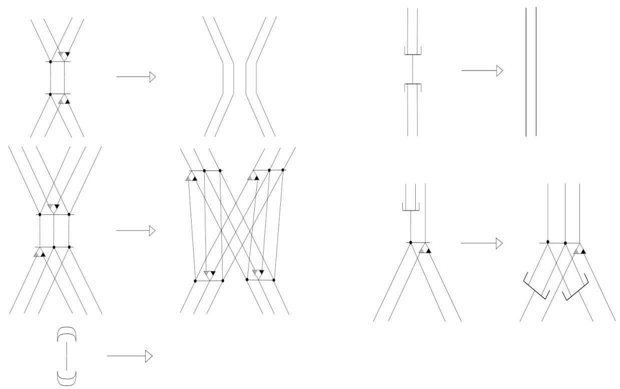

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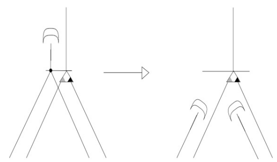

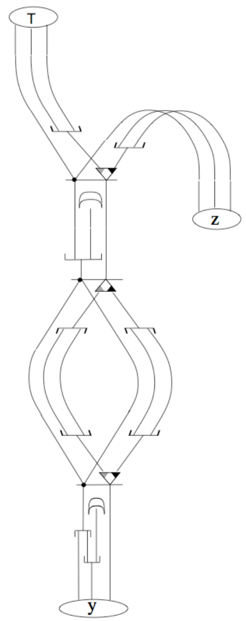

$(\lambda x.yx)z$

Implementation (5/5)

- beautiful Lamping's algorithm is unpractical
- highly exponential in the handling of fans node (not elementary recursive) [Asperti, Mairson 2000]
- nice algorithms unsharing paths to bound variables [Wadsworth 92, Shivers-Wand 2010]
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Speculative computations

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Permutations in call by value

- **Definition** `[call by value]`

  A **value** remains a value if computed or substituted by a value

  $$V \ ::= \ x \ | \ \lambda x.M$$

  The call-by-value reduction strategy is defined by:

  $$(\lambda x.M)V \xrightarrow[\text{cbv}]{} M\{x := V\}$$

  $$\frac{M \xrightarrow[\text{cbv}]{} M'}{MN \xrightarrow[\text{cbv}]{} M'N} \qquad \frac{N \xrightarrow[\text{cbv}]{} N'}{MN \xrightarrow[\text{cbv}]{} MN'}$$

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  Equivalence by permutations only permute disjoint redexes.

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- **Definition** [speculative call, Boudol-Petri 2010]

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Assigning meaning to λ-expressions

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Semantics

Definition A semantics of the λ-calculus is any equivalence such that:

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Böhm's theorem

Theorem [Bohm, 68]

Let M and N be 2 distinct normal forms. Then for any x and y, there exists a context $C[\,]$ such that:

$$C[M] \xrightarrow{*} x \quad \text{and} \quad C[N] \xrightarrow{*} y$$

Corollary Any (consistent) semantics of the λ-calculus cannot identify 2 distinct normal forms.

Notice Distinct normal forms means not η-interconvertible.

Exercice Bohm's thm for $I = \lambda x.x$ and $K = \lambda x.\lambda y.x$.

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Terms without normal forms

Lemma It is inconsistent to identify all terms without normal forms

Proof: Take $M = xa\Omega,\ N = y\Omega b$ where $\Omega = (\lambda x.xx)(\lambda x.xx)$

Let $C[\,] = (\lambda x.\lambda y.[\,])K(KI)$

Then $C[M] \xrightarrow{*} a$ and $C[N] \xrightarrow{*} b$

Question Which terms can be consistently identified ?

Easy terms `[Bohm, Jacopini]` $I = \Omega$ is consistent !

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Fact: Ω is totally undefined.
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Exercice:
Find other terms totally undefined. Try with $\Delta_3 = \lambda x.xxx$, $K = \lambda x.\lambda y.x$ and $Y = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx))$.

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Bohm trees (1/3)

Definition [72] The Bohm tree $\mathrm{BT}(M)$ of M is defined(?) as follows:

(1) If M has no hnf, $\mathrm{BT}(M) = \perp$

(2) If $M \xrightarrow{*} \lambda x_1.\lambda x_2 \ldots \lambda x_m.xM_1 M_2 \ldots M_n$, then

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Exercices Compute $\mathrm{BT}(I)$, $\mathrm{BT}(K)$, $\mathrm{BT}(\Omega)$, $\mathrm{BT}(Y)$, ...
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Theorem [74] Let $M \equiv_{BT} N$ iff $BT(M) = BT(N)$. Then $\equiv_{BT}$ is a (consistent) semantics of the λ-calculus.

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