Reductions and Causality (V)

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Plan

- a labeled λ-calculus
- lattice of labeled reductions
- labels and redex families
- canonical representatives
- strong normalization
- Hyland-Wadsworth labeled calculus
- labels and types

Labeled λ-calculus

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A labeled lambda-calculus (1/3)

- Give names to every redex and try make this naming consistent with permutation equivalence.

- Give names to some subterms:

$$M, N, \ldots ::= x \mid MN \mid \lambda x.M \mid M^{\alpha}$$

- Conversion rule is:

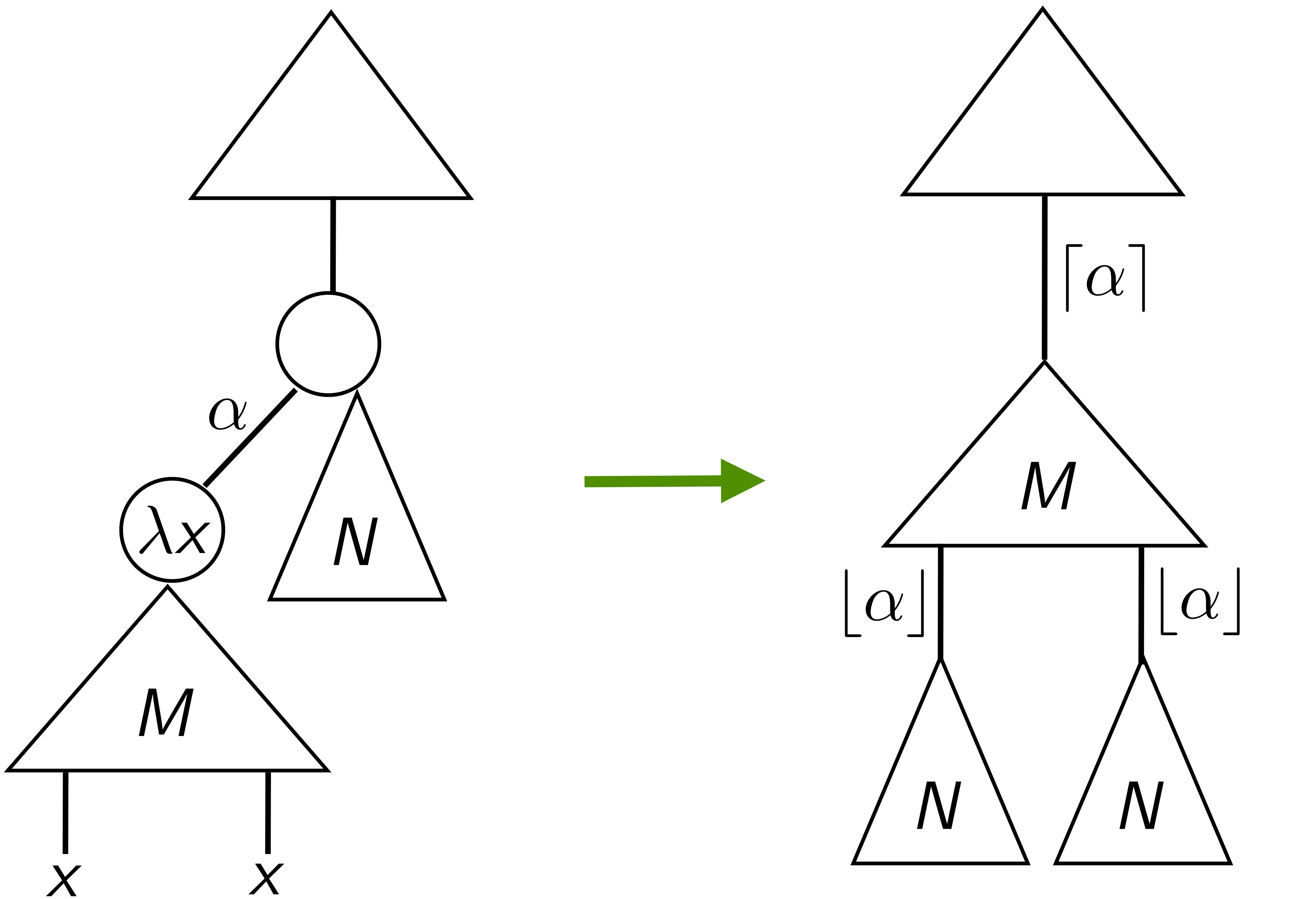
$$(\lambda x.M)^{\alpha} N \longrightarrow M^{\lceil \alpha \rceil}\{x := N^{\lfloor \alpha \rfloor}\}$$

α is **name** of redex

where

$$(M^{\alpha})^{\beta} = M^{\alpha\beta} \text{ and } M^{\alpha}\{x := N\} = (M\{x := N\})^{\alpha}$$

A labeled lambda-calculus (2/3)



A labeled lambda-calculus (3/3)

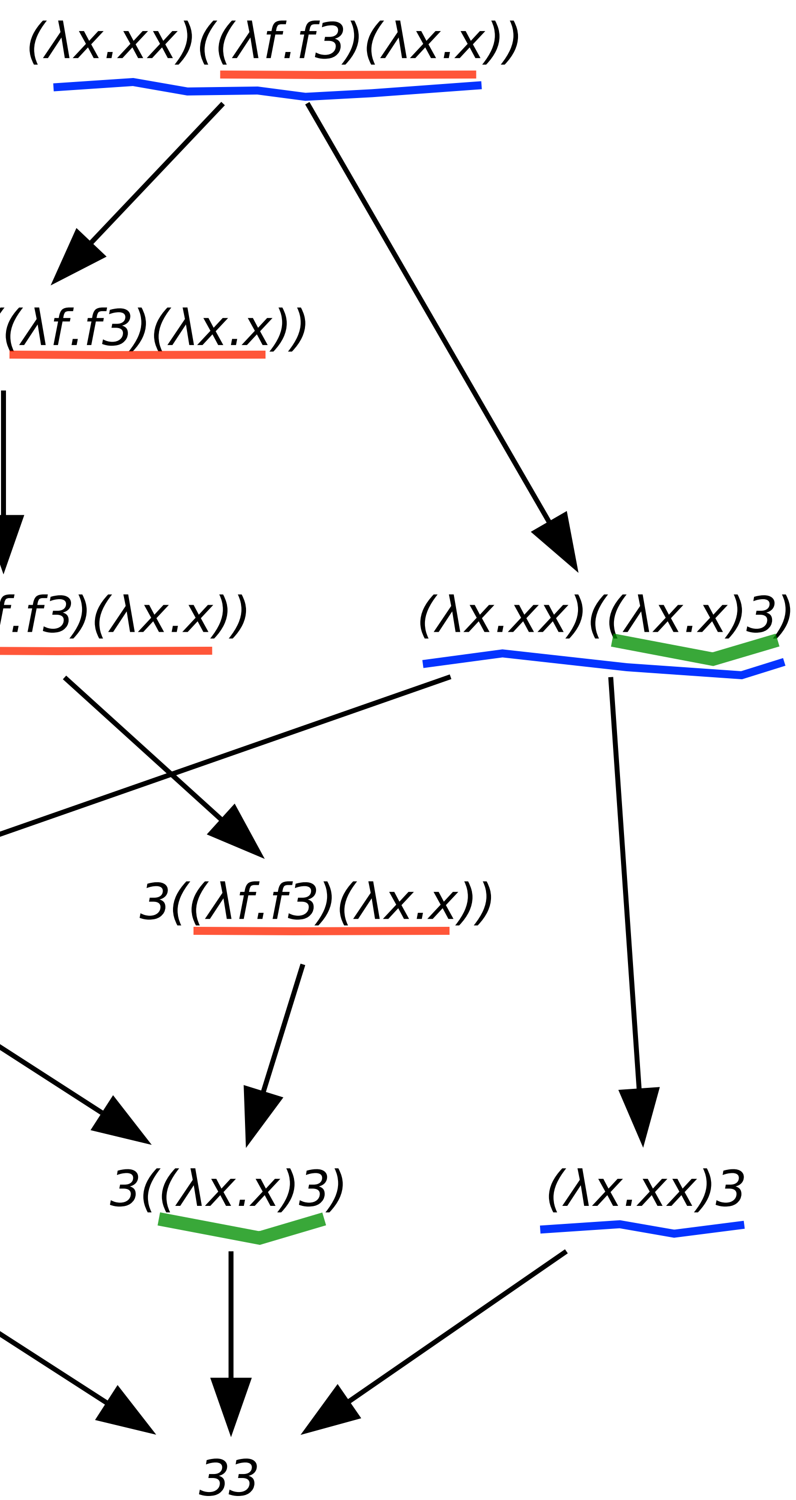
- Labels are strings of atomic labels:

$$\alpha, \beta, \ldots \quad ::= \quad \underbrace{a, b, c, \ldots \mid \lceil\alpha\rceil \mid \lfloor\alpha\rfloor}_{\texttt{atomic labels}} \mid \alpha\beta \mid \epsilon$$

- Labels are strings of atomic labels:

$a, b, c, \ldots$	atomic letters
$\lceil\alpha\rceil, \lfloor\alpha\rfloor, \ldots$	overlined, underlined labels
$\alpha\beta$	compound labels
$\epsilon = \lfloor\epsilon\rfloor = \lceil\epsilon\rceil$	empty label

Our favorite example



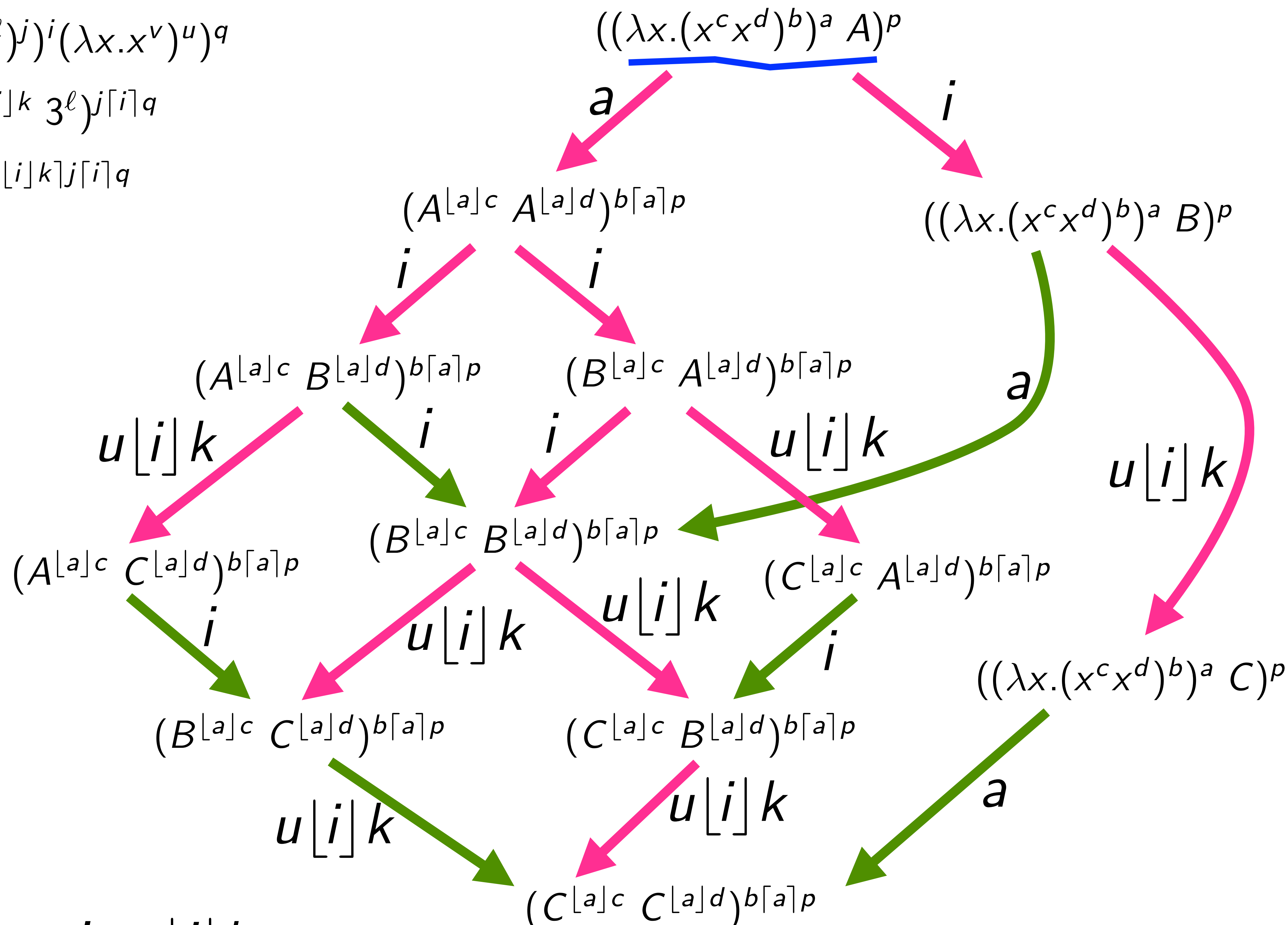
- 3 redex families: **red**, **blue**, **green**.

Our favorite example

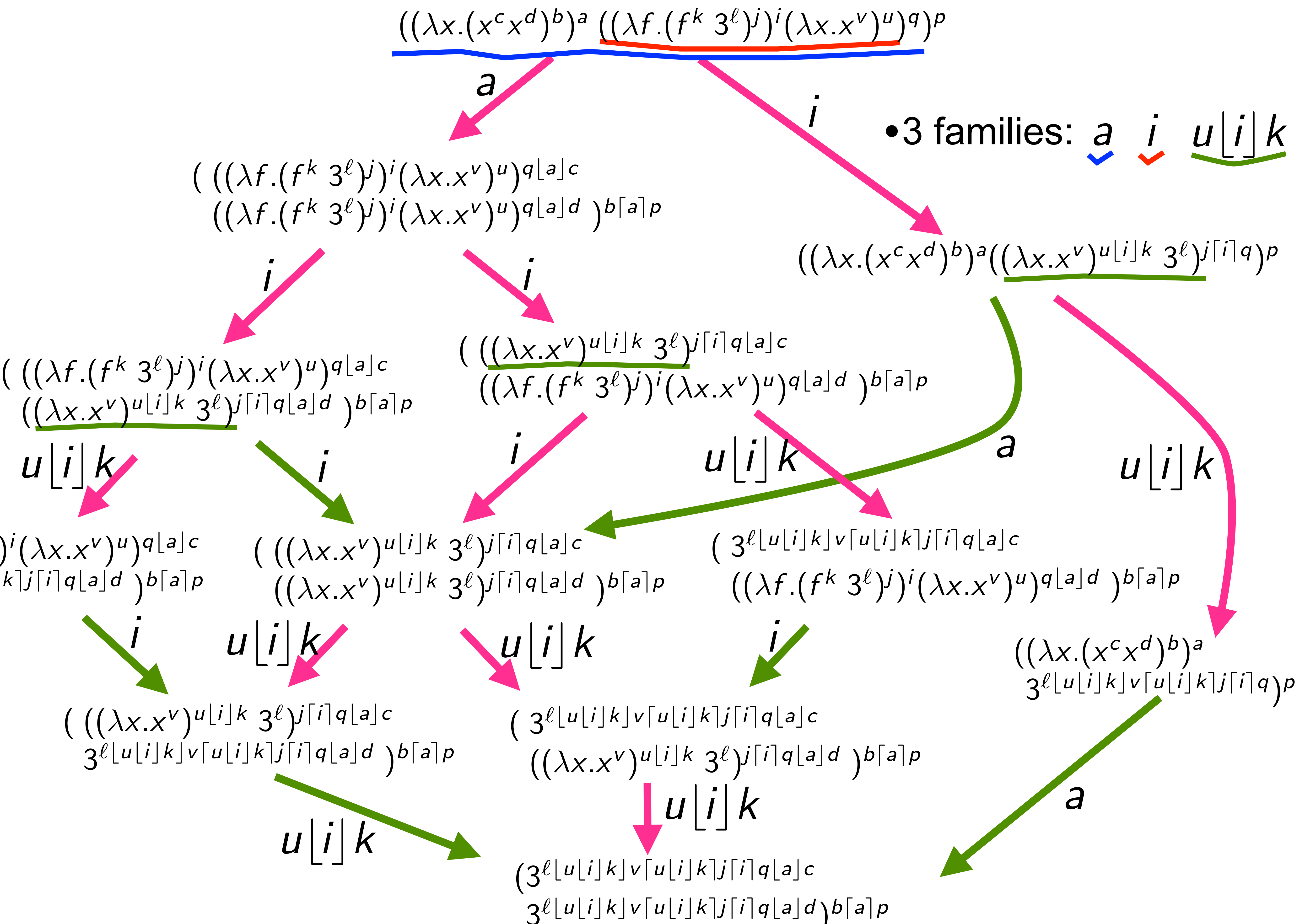
$A = ((\lambda f.(f^k\ 3^\ell)^j)^i(\lambda x.x^v)^u)^q$

$B = ((\lambda x.x^v)^{u\lfloor i\rfloor k}\ 3^\ell)^{j\lceil i\rceil q}$

$C = 3^{\ell\lfloor u\lfloor i\rfloor k\rfloor v\lceil u\lfloor i\rfloor k\rceil j\lceil i\rceil q}$



- 3 families: a i $u\lfloor i\rfloor k$



$((\lambda x.(x^c x^d)^b)^a\ ((\lambda f.(f^k\ 3^\ell)^j)^i(\lambda x.x^v)^u)^q)^p$
a
i
•3 families: a i $u\lfloor i\rfloor k$
$(\ ((\lambda f.(f^k\ 3^\ell)^j)^i(\lambda x.x^v)^u)^{q\lfloor a\rfloor c}$
$((\lambda f.(f^k\ 3^\ell)^j)^i(\lambda x.x^v)^u)^{q\lfloor a\rfloor d}\ )^{b\lceil a\rceil p}$
$((\lambda x.(x^c x^d)^b)^a((\lambda x.x^v)^{u\lfloor i\rfloor k}\ 3^\ell)^{j\lceil i\rceil q})^p$
i
i
$(\ ((\lambda f.(f^k\ 3^\ell)^j)^i(\lambda x.x^v)^u)^{q\lfloor a\rfloor c}$
$((\lambda x.x^v)^{u\lfloor i\rfloor k}\ 3^\ell)^{j\lceil i\rceil q\lfloor a\rfloor d}\ )^{b\lceil a\rceil p}$
$(\ ((\lambda x.x^v)^{u\lfloor i\rfloor k}\ 3^\ell)^{j\lceil i\rceil q\lfloor a\rfloor c}$
$((\lambda f.(f^k\ 3^\ell)^j)^i(\lambda x.x^v)^u)^{q\lfloor a\rfloor d}\ )^{b\lceil a\rceil p}$
$u\lfloor i\rfloor k$
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$u\lfloor i\rfloor k$
a
$u\lfloor i\rfloor k$
$)^i(\lambda x.x^v)^u)^{q\lfloor a\rfloor c}$
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$((\lambda f.(f^k\ 3^\ell)^j)^i(\lambda x.x^v)^u)^{q\lfloor a\rfloor d}\ )^{b\lceil a\rceil p}$
i
$u\lfloor i\rfloor k$
$u\lfloor i\rfloor k$
i
$((\lambda x.(x^c x^d)^b)^a$
$3^{\ell\lfloor u\lfloor i\rfloor k\rfloor v\lceil u\lfloor i\rfloor k\rceil j\lceil i\rceil q})^p$
$(\ ((\lambda x.x^v)^{u\lfloor i\rfloor k}\ 3^\ell)^{j\lceil i\rceil q\lfloor a\rfloor c}$
$3^{\ell\lfloor u\lfloor i\rfloor k\rfloor v\lceil u\lfloor i\rfloor k\rceil j\lceil i\rceil q\lfloor a\rfloor d}\ )^{b\lceil a\rceil p}$
$(\ 3^{\ell\lfloor u\lfloor i\rfloor k\rfloor v\lceil u\lfloor i\rfloor k\rceil j\lceil i\rceil q\lfloor a\rfloor c}$
$((\lambda x.x^v)^{u\lfloor i\rfloor k}\ 3^\ell)^{j\lceil i\rceil q\lfloor a\rfloor d}\ )^{b\lceil a\rceil p}$
$u\lfloor i\rfloor k$
a
$u\lfloor i\rfloor k$
$(3^{\ell\lfloor u\lfloor i\rfloor k\rfloor v\lceil u\lfloor i\rfloor k\rceil j\lceil i\rceil q\lfloor a\rfloor c}$
$3^{\ell\lfloor u\lfloor i\rfloor k\rfloor v\lceil u\lfloor i\rfloor k\rceil j\lceil i\rceil q\lfloor a\rfloor d})^{b\lceil a\rceil p}$

Creation of redexes (1/3)

$$
\begin{tikzcd}[column sep=small]
& ((\lambda f.(f^c\ 3^d)^b)^a\ ((\lambda x.x^v)^u(\lambda x.(x^k x^\ell)^j)^i\,)^q\ )^p \arrow[dl, "a"'] \arrow[dr, "u"] & \\
(((\lambda x.x^v)^u(\lambda x.(x^k x^\ell)^j)^i)^{q\lfloor a\rfloor c}\ 3^d\ )^{b\lceil a\rceil p} \arrow[dr, "u"'] & & ((\lambda f.(f^c\ 3^d)^b)^a\ (\lambda x.(x^k x^\ell)^j)^{i\lfloor u\rfloor v\lceil u\rceil q}\ )^p \arrow[dl, "a"] \\
& ((\lambda x.(x^k x^\ell)^j)^{i\lfloor u\rfloor v\lceil u\rceil q\lfloor a\rfloor c}\ 3^d)^{b\lceil a\rceil p} &
\end{tikzcd}
$$

- 3 families: a u $i\lfloor u\rfloor v\lceil u\rceil q\lfloor a\rfloor c$
- 2 independent redexes *a* and *u* creates the new one

Creation of redexes (2/3)

$$
\begin{tikzcd}[column sep=small]
& \big(\,(\lambda f.(f^c\, 3^d)^b)^a\, ((\lambda x.(\lambda y.x^k)^j)^i I)^q\,\big)^p \arrow[dl, "a"'] \arrow[dr, "i"] & \\
\big(\,((\lambda x.(\lambda y.x^k)^j)^i I)^{q\lfloor a\rfloor c}\, 3^d\big)^{b\lceil a\rceil p} \arrow[dr, "i"] & & \big(\,(\lambda f.(f^c\, 3^d)^b)^a (\lambda y.I^{\lfloor i\rfloor k})^{j\lceil i\rceil q}\,\big)^p \arrow[dl, "a"'] \\
& ((\lambda y.I^{\lfloor i\rfloor k})^{j\lceil i\rceil q\lfloor a\rfloor c}\, 3^d)^{b\lceil a\rceil p} &
\end{tikzcd}
$$

- 3 families: a i $j\lceil i\rceil q\lfloor a\rfloor c$
- 2 independent redexes *a* and *u* creates the new one

Creation of redexes (3/3)

$((\lambda x.(x^c x^d)^b)^a\ \Delta)^p \qquad\qquad \Delta = ((\lambda x.(x^g x^h)^f)^e$

$(\Delta^{\alpha_1}\Delta^{\alpha'_1})^{\beta_1} = (\Delta^{\lfloor a \rfloor c}\Delta^{\lfloor a \rfloor d})^{b\lceil a \rceil p}$

$(\Delta^{\alpha_2}\Delta^{\alpha'_2})^{\beta_2} = (\Delta^{\lfloor e\lfloor a \rfloor c\rfloor g}\Delta^{\lfloor e\lfloor a \rfloor c\rfloor h})^{f\lceil e\lfloor a \rfloor c\rceil b\lceil a \rceil p}$

$(\Delta^{\alpha_3}\Delta^{\alpha'_3})^{\beta_3} = (\Delta^{\lfloor e\lfloor e\lfloor a \rfloor c\rfloor g\rfloor g}\Delta^{\lfloor e\lfloor e\lfloor a \rfloor c\rfloor g\rfloor h})^{f\lceil e\lfloor e\lfloor a \rfloor c\rfloor g\rceil f\lceil e\lfloor a \rfloor c\rceil b\lceil a \rceil p}$

$(\Delta^{\alpha_{n+1}}\Delta^{\alpha'_{n+1}})^{\beta_{n+1}} = (\Delta^{\lfloor e\alpha_n \rfloor g}\Delta^{\lfloor e\alpha_n \rfloor h})^{f\lceil e\alpha_n \rceil \beta_n}$

- infinite number of families

Permutation equivalence (1/7)

- **Proposition** [residuals of labeled redexes]
  $S \in R/\rho$ implies $\mathrm{name}(R) = \mathrm{name}(S)$

- **Definition** [created redexes] Let $\langle \rho, R \rangle$ be historical redex.
  We say that ρ **creates** R when $\nexists R',\ R \in R'/\rho$.

- **Proposition** [created labeled redexes]
  If S creates R, then $\mathrm{name}(S)$ is strictly contained in $\mathrm{name}(R)$.

Permutation equivalence (2/7)

Proof (cont'd) Created redexes contains names of creator

$$\underbrace{(\lambda x.\cdots(x^{\beta}\,N)\cdots)^{\alpha}\,(\lambda y.M)^{\gamma}}_{\alpha} \to \cdots\underbrace{((\lambda y.M)^{\gamma\lfloor\alpha\rfloor\beta}\,N')}_{\gamma\lfloor\alpha\rfloor\beta}\cdots$$

creates

$$\underbrace{((\lambda x.(\lambda y.M)^{\gamma})^{\alpha}\,N)^{\beta}}_{\alpha}\,P \to \underbrace{(\lambda y.M')^{\gamma\lceil\alpha\rceil\beta}\,P}_{\gamma\lceil\alpha\rceil\beta}$$

creates

$$\underbrace{((\lambda x.\,x^{\gamma})^{\alpha}\,(\lambda y.M)^{\delta})^{\beta}}_{\alpha}\,N \to \underbrace{(\lambda y.M)^{\delta\lfloor\alpha\rfloor\gamma\lceil\alpha\rceil\beta}\,N}_{\delta\lfloor\alpha\rfloor\gamma\lceil\alpha\rceil\beta}$$

creates

Permutation equivalence (3/7)

- **Labeled laws** $M^\alpha \{x := N\} = (M\{x := N\})^\alpha \qquad (M^\alpha)^\beta = M^{\alpha\beta}$

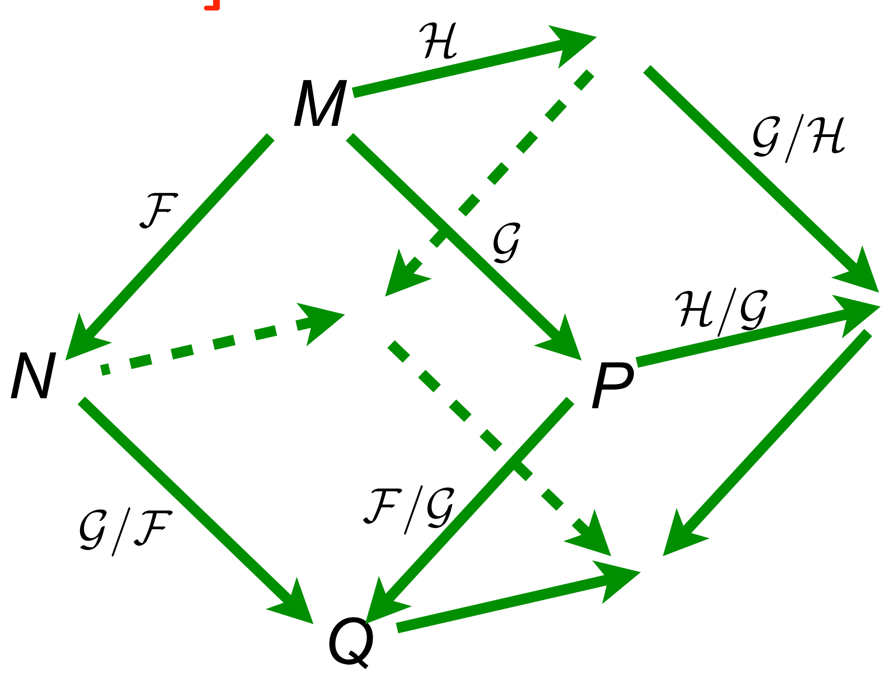
  If $M \longrightarrow N$, then $M^\alpha \longrightarrow N^\alpha$

- **Labeled parallel moves lemma+** [74]

  If $M \xrightarrow{\mathcal{F}} N$ and $M \xrightarrow{\mathcal{G}} P$, then $N \xrightarrow{\mathcal{G}/\mathcal{F}} Q$ and $P \xrightarrow{\mathcal{F}/\mathcal{G}} Q$ for some Q.

- **Parallel moves lemma++** [The Cube Lemma]

  still holds.



Permutation equivalence (4/7)

- Labels do not break Church-Rosser, nor residuals

- Labels refine λ-calculus:
  - any unlabeled reduction can be performed in the labeled calculus
  - but two cofinal unlabeled reductions may no longer be cofinal

  Take $I(I3)$ with $I = \lambda x.x$.

$$\begin{tikzcd}[column sep=tiny]
 & ((\lambda x.x^{c})^{b}\ ((\lambda x.x^{f})^{e}\ 3^{g})^{d})^{a} \arrow[dl] \arrow[dr] & \\
((\lambda x.x^{f})^{e}\ 3^{g})^{d\lfloor b\rfloor c\lceil b\rceil a} & & ((\lambda x.x^{c})^{b}\ 3^{g\lfloor e\rfloor f\lceil e\rceil d})^{a}
\end{tikzcd}$$

Permutation equivalence (5/7)

- **Definition** `[pure labeled calculus]`

Pure labeled terms are labeled terms where all subterms have non empty labels.

- **Theorem** `[labeled permutation equivalence, 76]`

Let ρ and σ be coinitial pure labeled reductions.
Then $\rho \simeq \sigma$ iff ρ and σ are labeled cofinal.

Proof Let $\rho \simeq \sigma$. Then obvious because of labeled parallel moves lemma.

Conversely, we apply standardization thm and following lemma.

- **Lemma** `[uniqueness of pure labeled standard reductions]`

Proof ...

Permutation equivalence (6/7)

Proof [uniqueness of labeled standard]

Let ρ and σ be 2 distinct coinitial pure labeled standard reductions.

Take first step when they diverge. Call M that term.
We make structural induction on M. Say ρ is more to the left.
If first step of ρ contracts an internal redex, we use induction.
If first step of ρ contracts an external redex, then:

$$\begin{tikzcd}
 & M = ((\lambda x.P)^{\alpha}\, Q)^{\beta} \arrow[dl] \arrow[ddrr, bend left, "\textbf{st}"] & & \\
P^{\lceil\alpha\rceil\beta}\{x := Q^{\lfloor\alpha\rfloor}\} \arrow[dr, "\textbf{st}"] & & & \\
 & N^{\lceil\alpha\rceil\beta} & \neq & ((\lambda x.A)^{\alpha} B)^{\beta}
\end{tikzcd}$$

Permutation equivalence (7/7)

- **Corollary** `[labeled prefix ordering]`

  Let $\rho : M \xrightarrow{\star} N$ and $\sigma : M \xrightarrow{\star} P$ be coinitial pure labeled reductions. Then $\rho \sqsubseteq \sigma$ iff $N \xrightarrow{\star} P$.

- **Corollary** `[lattice of labeled reductions]`

  Labeled reduction graphs are upwards semi lattices for any pure labeling.

- **Exercice** Try on $(\lambda x.x)((\lambda y.(\lambda x.x)a)b)$ or $(\lambda x.xx)(\lambda x.xx)$

Redex families

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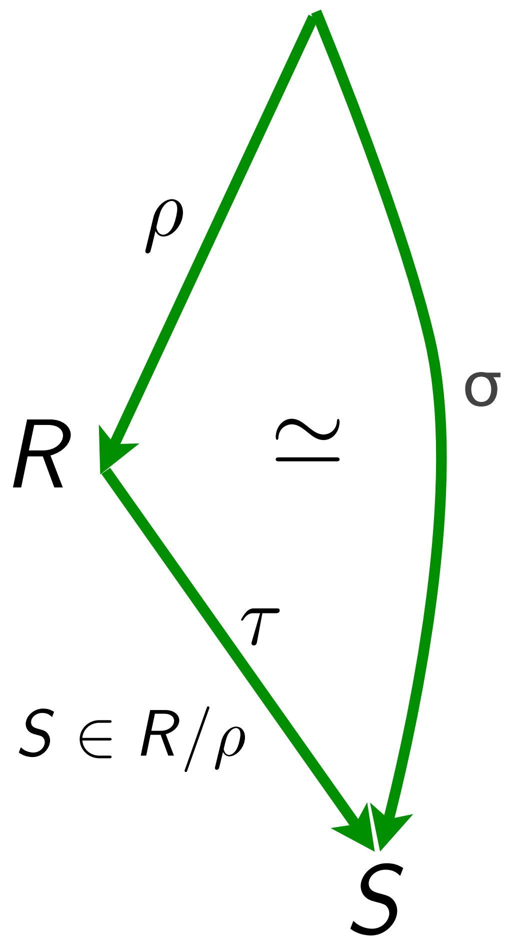
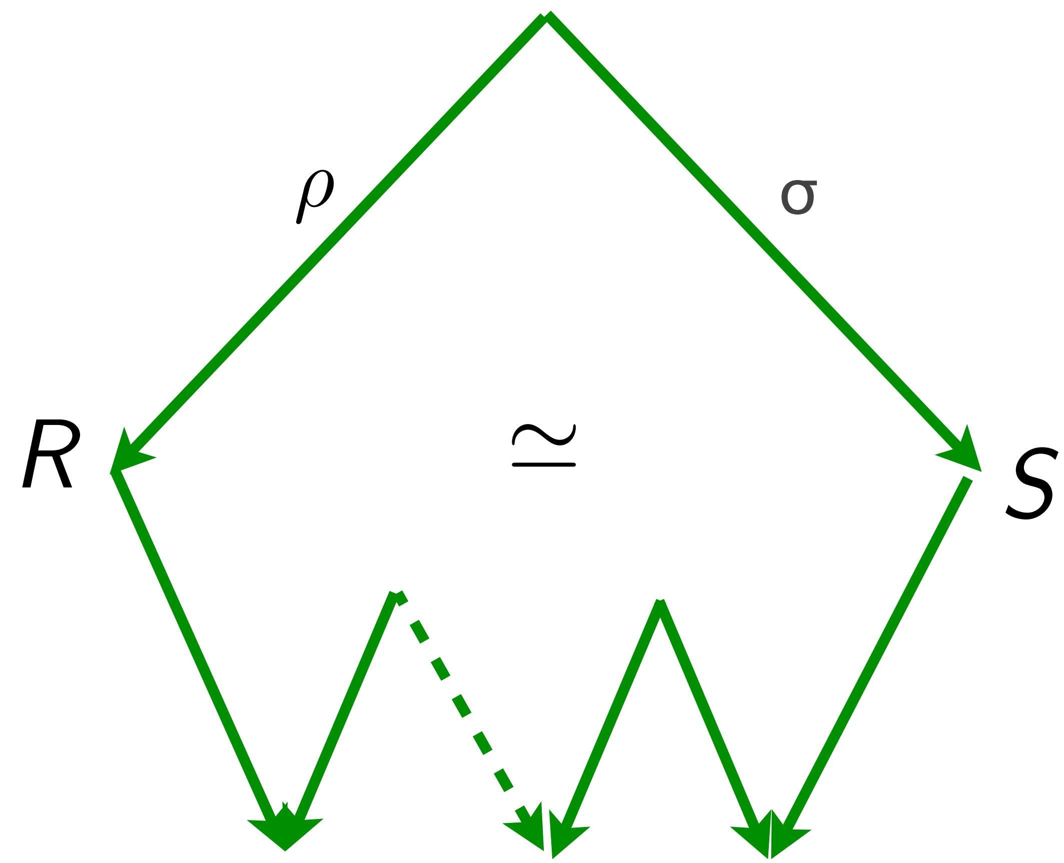
Labels and history (1/4)





$\langle \rho, R \rangle \leq \langle \sigma, S \rangle$

$\langle \rho, R \rangle \sim \langle \sigma, S \rangle$

$\mathsf{name}(R) = \mathsf{name}(S)$

Labels and history (2/4)

- **Proposition** `[same history → same name]`

  In the labeled λ-calculus, for any labeling, we have:

  $\langle \rho, R \rangle \sim \langle \sigma, S \rangle$ implies $\mathrm{name}(R) = \mathrm{name}(S)$

- The opposite direction is clearly not true for any labeling

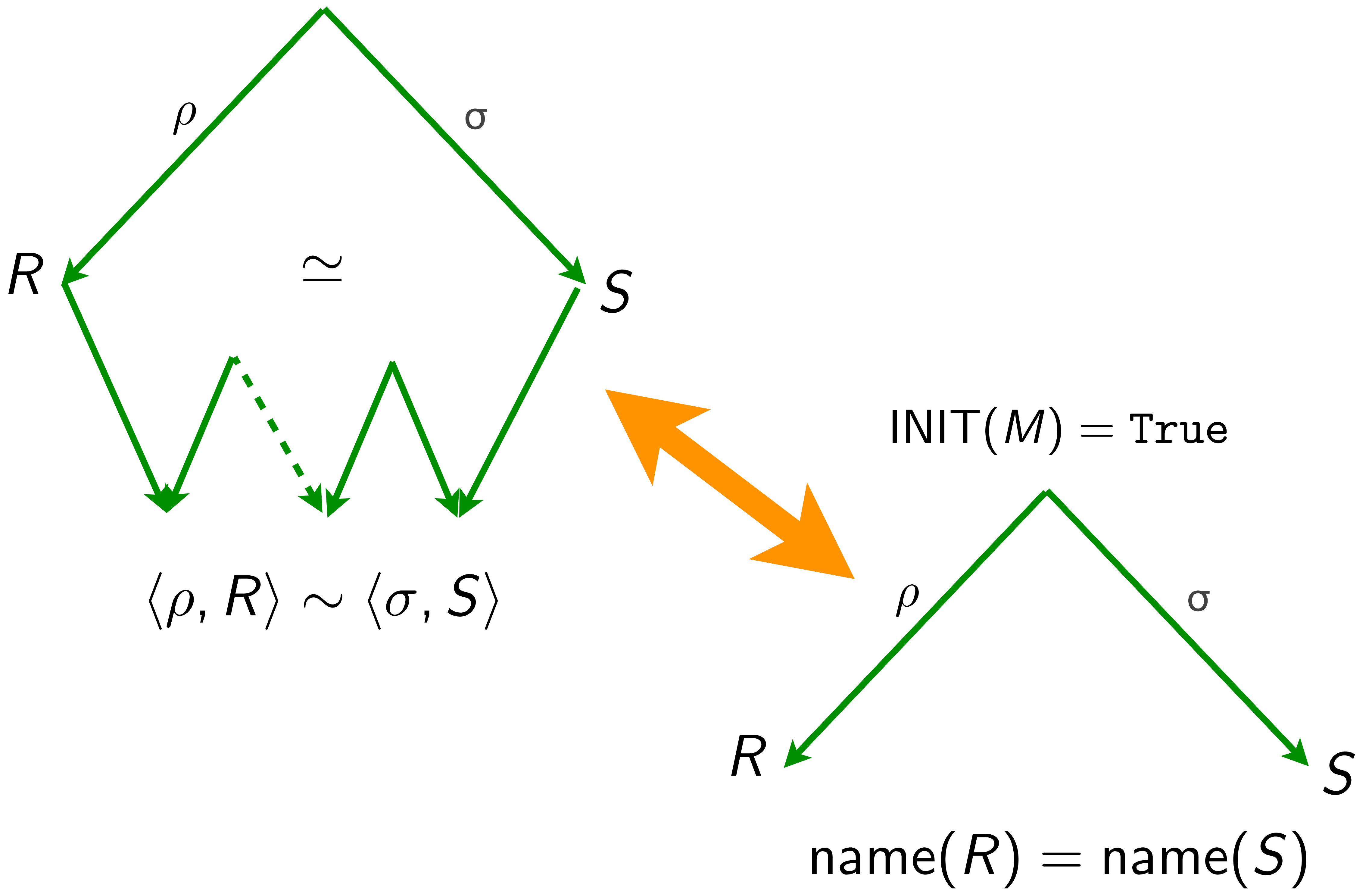
  (For instance, take all labels equal)

- But it is true when all labels are distinct atomic letters in the initial term.

- **Definition** `[all labels distinct letters]`

  $\mathrm{INIT}(M) = \mathtt{True}$ when all labels in M are distinct letters.

Labels and history (3/4)



Labels and history (4/4)

- **Theorem** `[same history = same name, 76]`

  When INIT(M) and reductions ρ and σ start from M :

  $$\langle \rho, R \rangle \sim \langle \sigma, S \rangle \quad \text{iff} \quad \mathsf{name}(R) = \mathsf{name}(S)$$

- **Corollary** `[decidability of family relation]`

  The family relation is decidable (although complexity is proportional to length of standard reduction).

Finite developments

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Parallel steps revisited (1/3)

- parallel steps were defined with inside-out strategy
  `[a la Martin-Löf]`

- can we take any order as reduction strategy ?

- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redexes in M is any reduction contracting only residuals of $\mathcal{F}$.
  A **development** of $\mathcal{F}$ is any maximal relative reduction of $\mathcal{F}$.

Parallel steps revisited (2/3)

- **Theorem** `[Finite Developments, Curry, 50]`

  Let $\mathcal{F}$ be set of redexes in M.

  (1) there are no infinite relative reductions of $\mathcal{F}$,
  (2) they all finish on same term N
  (3) Let R be redex in M. Residuals of R by all finite developments of $\mathcal{F}$ are the same.

- Similar to parallel moves lemma, but we considered particular inside-out reduction strategy.

Parallel steps revisited (3/3)

- **Notation'** `[parallel reduction steps]`
  Let $\mathcal{F}$ be set of redexes in M. We write $M \xrightarrow{\mathcal{F}} N$
  if a development of $\mathcal{F}$ connects M to N.

- This notation is consistent with previous results

- Corollaries of FD thm are also parallel moves + cube lemmas

Finite and infinite reductions (1/3)

- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redex families is any reduction contracting redexes in families of $\mathcal{F}$.

  A **development** of $\mathcal{F}$ is any maximal relative reduction.

- **Theorem** `[Finite Developments+, 76]`
  Let $\mathcal{F}$ be a finite set of redex families.

  (1) there are no infinite reductions relative to $\mathcal{F}$,
  (2) they all finish on same term N
  (3) All developments are equivalent by permutations.

Finite and infinite reductions (2/3)

- **Corollary** An **infinite reduction** contracts an **infinite set of redex families.**

- **Corollary** The first-order typed λ-calculus strongly terminates.

Proof In first-order typed λ-calculus:

(1) residuals $R' = (\lambda x.M')N'$ of $R = (\lambda x.M)N$ keep the same type of the function part

(2) new redexes have lower type of their function part

Finite and infinite reductions (3/3)

Proof (cont'd) Created redexes have lower type

$$\underbrace{(\lambda x.\cdots xN\cdots)}_{\sigma\to\tau}\underbrace{(\lambda y.M)}_{\sigma} \longrightarrow \cdots\underbrace{(\lambda y.M)}N'\cdots$$

creates

$$\underbrace{(\lambda x.\underbrace{\lambda y.M}_{\tau})}_{\sigma\to\tau}NP \longrightarrow \underbrace{(\lambda y.M')}_{\tau}P$$

creates

$$\underbrace{(\lambda x.x)}_{\sigma\to\sigma}\underbrace{(\lambda y.M)}_{\sigma}N \longrightarrow \underbrace{(\lambda y.M)}_{\sigma}N$$

creates

Inside-out reductions

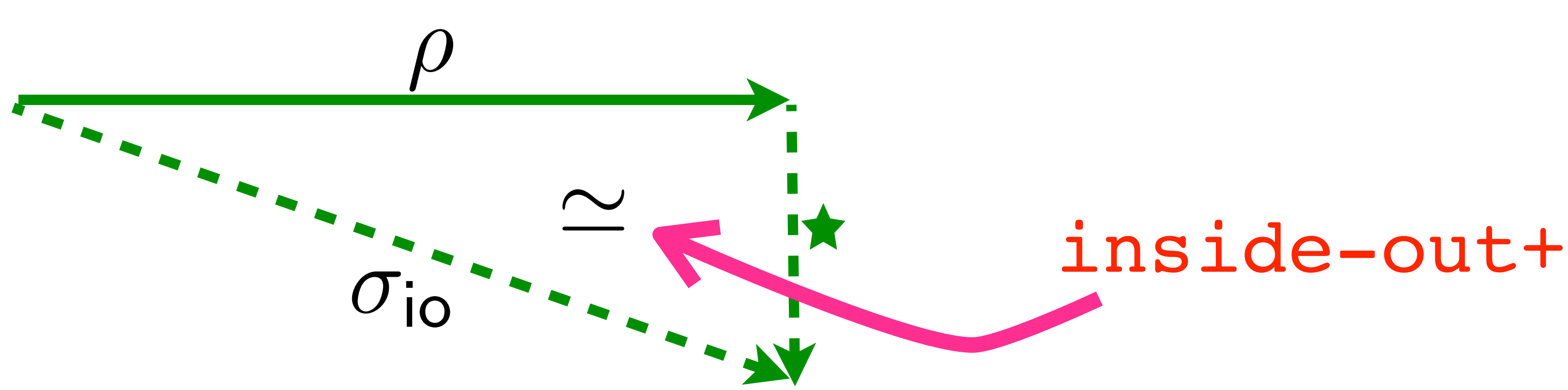
- **Definition:** The following reduction is **inside-out**

$$\rho : M = M_0 \xrightarrow{R_1} M_1 \xrightarrow{R_2} M_2 \cdots \xrightarrow{R_n} M_n = N$$

iff for all i and j, $i < j$, then R_j is not residual along ρ of some R'_j inside R_i in M_{i-1}.

- **Theorem** [Inside-out completeness, 74]

Let $M \xrightarrow{\star} N$. Then $M \xrightarrow[\textbf{io}]{\star} P$ and $N \xrightarrow{\star} P$ for some P.



Exercices

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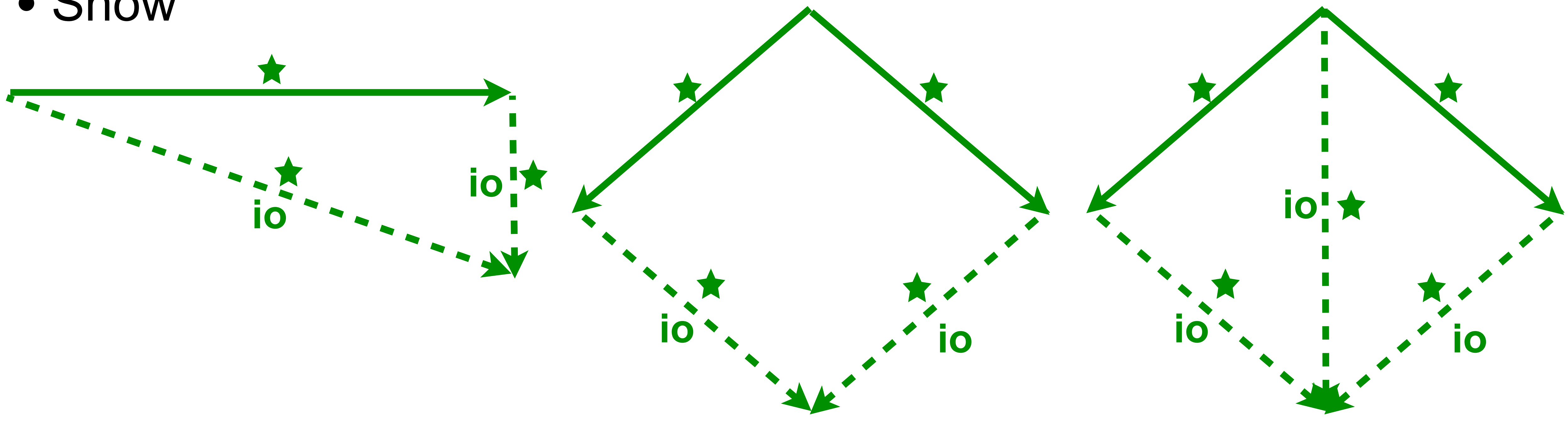
Exercices

- Show



Strong normalization

Strong normalization (1/3)

- Another labeled λ-calculus was considered to study Scott D-infinity model `[Hyland-Wadsworth, 74]`

- D-infinity projection functions on each subterm (n is any integer):

$$M, N, \ldots ::= x^n \mid (MN)^n \mid (\lambda x.M)^n$$

- Conversion rule is:

$$((\lambda x.M)^{n+1} N)^p \longrightarrow M\{x := N_{[n]}\}_{[n][p]}$$

$n+1$ is **degree** of redex

$$U_{[m][n]} = U_{[p]} \quad \text{where} \quad p = \min\{m, n\}$$

$$x^n\{x := M\} = M_{[n]}$$

Strong normalization (2/3)

- **Proposition** Hyland-Wadsworth calculus is derivable from labeled calculus by simple homomorphism on labels.

  Proof Assign an integer to any atomic letter and take:

  $$\mathsf{h}(\alpha\beta) = \mathsf{min}\{\mathsf{h}(\alpha),\ \mathsf{h}(\beta)\}$$

  $$\mathsf{h}(\lceil\alpha\rceil) = \mathsf{h}(\lfloor\alpha\rfloor) = \mathsf{h}(\alpha) - 1$$

- **Proposition** Hyland-Wadsworth calculus strongly normalizes.

- **Corollary** When only a finite set of redex degrees is contracted, there is strong normalization.