1. Consider the term \( P = \nu y.(x(x), x(x), \nu y.(\nu y.(\nu x.(\nu x.(\nu x.(x(x))))) \) with \( x \neq y \).

(a) What is the set of free names of \( P \)?

Solution: \( \{x\} \).

(b) Define a process \( P' \) that is \( \alpha \)-equivalent to \( P \) and has all bound names distinct from each other and from all free names.

Solution: \( P' = \nu y_1.(x(x_1), x_1(x_2), \nu x_2.(\nu x_3.(\nu x_4.(x_4))) \).

(c) What is the set of free names of \( P' \)?

Solution: \( \{x\} \), since \( \alpha \)-conversion doesn’t change the free names.

(d) Show the sequence of three reduction steps (\( \longrightarrow \)) starting at \( P' \), taking care to make explicit any scope extrusions you need (i.e. use of \( \text{str-ex} \) in \( \equiv \)).

Solution:

\[
P' \\
\equiv \nu y_1, y_2.(x(x_1), x_1(x_2), \nu x_2.(\nu x_3.(\nu x_4.(x_4)))) \text{ extrusion of } y_2 \\
\longrightarrow \nu y_1, y_2.(y_2(x_2), \nu x_2.(\nu x_3.(\nu x_4.(x_4)))) \text{ communication on } x \\
\equiv \nu y_1, y_2, x_3.(y_2(x_2), \nu x_2.(\nu x_3.(\nu x_4.(x_4)))) \text{ extrusion of } x_3 \\
\longrightarrow \nu y_1, y_2, x_3.(\nu x_3(y_1), x_3(x_4)) \text{ communication on } y_2 \\
\longrightarrow \nu y_1, y_2, x_3.(0 | 0) \text{ communication on } x_3
\]

2. This question explores the relationship between name hiding and bisimulation in the core \( \pi \)-calculus (i.e. the calculus on slide 3 with no extra features).

(a) Prove that strong bisimilarity is closed by new binding, i.e. \( P \sim Q \) implies \( \nu x.P \sim \nu x.Q \). You may only use the basic definitions of bisimulation and labelled transition (no “up to” techniques).

Hint: start with a relation \( R = \{\nu x.P, \nu x.Q / P \sim Q\} \) and try to show that \( R \) is a strong bisimulation. You may have to add some more pairs to \( R \).

Solution: Take \( R' = R \cup (~) \). We aim to show that \( R' \) is a bisimulation. To do that we need to consider \( (P_0, Q_0) \in R' \) and \( P_0 \xrightarrow{\alpha} P_0' \) with \( \text{bn}(\alpha) \cap \text{fn}(Q_0) = \emptyset \).

We distinguish two cases.

Case \( P_0 \sim Q_0 \): By definition of \( \sim \), there exists \( Q_0' \) such that \( Q_0 \xrightarrow{\alpha} Q_0' \) and \( P_0' \sim Q_0' \), hence \( (P_0', Q_0') \in R' \), as desired.

Case \( (P_0, Q_0) \in R \): Then there exists \( P \) and \( Q \) such that \( P \sim Q \) and \( Q_0 = \nu x.Q \). We consider the two possible ways that the labelled transition \( P_0 \xrightarrow{\alpha} P_0' \) could have been derived.

Case (lab-new): Then there exists \( P \) such that \( P \xrightarrow{\alpha} P' \) and \( P_0' = \nu x.P' \) and \( x \notin \text{bn}(\alpha) \). By hypothesis, we also have \( \text{bn}(\alpha) \cap \text{fn}(Q_0) = \emptyset \), hence \( \text{bn}(\alpha) \cap \text{fn}(Q) = \emptyset \), thus it is safe to apply the definition of bisimulation to the hypothesis \( P \sim Q \); hence there exists \( Q' \) such that \( Q \xrightarrow{\alpha} Q' \) and \( P' \sim Q' \). Let \( Q_0' = \nu x.Q' \). Then by (lab-new), \( Q_0 \xrightarrow{\alpha} Q_0' \). Finally, \( (P_0', Q_0') \in R \subseteq R' \), as desired.
Case (lab-open): Then there exists $P'$ and $w$ such that $P \xrightarrow{\overline{w}} P'$ and $w \neq x$ and $P'_0 = P'$ and $\alpha = \overline{w}(x)$. By hypothesis, $P \sim Q$, so there exists $Q'$ such that $Q \xrightarrow{\overline{w}} Q'$ and $P' \sim Q'$. Let $Q'_0 = Q'$. By (lab-open), $Q_0 \xrightarrow{\alpha} Q'_0$.

Finally, $(P'_0, Q'_0) \in (\sim) \subseteq R'$, as desired.

Since $R'$ is a symmetric relation, we conclude that it is a bisimulation. Hence for any $P \sim Q$, we have $(\nu x.P, \nu x.Q) \in R' \subseteq (\sim)$, which completes the proof.

(b) Give a counterexample to show that the converse is false, i.e. $\nu x.P \sim \nu x.Q$ does not imply $P \sim Q$.

Solution: Take $P = \pi y$ and $Q = 0$ with $x, y$ distinct. Then $\nu x.P \sim \nu x.Q$ since both sides are deadlocked. However $P \not\sim Q$ since $P$ has the labelled transition $\xrightarrow{\nu} Q$ but $Q$ does not.

(c) However if we hide and then reveal a name, then it is as if we never hid it! Prove that $\nu x.(\bar{k}x \mid P) \sim \nu x.(\bar{k}x \mid Q)$ implies $P \sim Q$ for $k \notin \text{fn}(P) \cup \text{fn}(Q) \cup \{x\}$.

Hint: There is no need to explicitly construct a bisimulation relation containing $(P, Q)$. Instead consider the $\bar{k}(x)$ labelled transitions of $\nu x.(\bar{k}x \mid P)$ and $\nu x.(\bar{k}x \mid Q)$. Take care to clearly write out any proof trees you use when deriving labelled transitions.

Solution: We can infer a bound output for $\nu x.(\bar{k}x \mid P)$ by the following derivation:

$$
\begin{array}{c}
\bar{k}x \\
\xrightarrow{\nu x \bar{k}x \mid P} 0 \\
\xrightarrow{P \bar{k}x} 0
\end{array}
$$

Since we have two bisimilar processes, $\nu x.(\bar{k}x \mid P) \sim \nu x.(\bar{k}x \mid Q)$, we know that the right-hand process must be able to match the transition we just derived, i.e. there exists $Q''$ such that $0 \mid P \sim Q''$ and $\nu x.(\bar{k}x \mid Q) \xrightarrow{\bar{k}(x)} Q''$.

There are only two possible rules that the this last labelled transition can be derived from, the first of which turns out to be impossible.

Case (lab-new): By the side condition for the rule, the only way it can be applied is if we do $\alpha$-conversion on $x$, i.e. we have a derivation of the form:

$$
\begin{array}{c}
\bar{k}x \mid \{x'/x\}Q \\
\xrightarrow{\nu x' \bar{k}x' \mid \{x'/x\}Q}
\end{array}
$$

where $x'$ is fresh. Suppose, for contradiction, that the premiss were derivable. By hypothesis, $k \notin \text{fn}(Q)$, hence $k \notin \text{fn}(\{x'/x\}Q)$, therefore we cannot have $\{x'/x\}Q \xrightarrow{\bar{k}(x)} Q''$. Nor can $\bar{k}x' \bar{k}(x)$ since $x'$ is free here, disallowing any bound output. Thus this case is impossible.

Case (lab-open): Then the premiss is $\bar{k}x \mid Q \xrightarrow{\bar{k}x} Q''$. Since $k \notin \text{fn}(Q)$, the output is due to $\bar{k}x \xrightarrow{\bar{k}x} 0$, thus $Q'' = 0 \mid Q$. Finally, $P \sim 0 \mid P \sim 0 \mid Q \sim Q$, as desired.

3. This question addresses relationships between labelled transitions and bars in the core $\pi$-calculus.

(a) Prove that $P \xrightarrow{\pi y} P'$ implies $P \mid x$. Hint: induct on the derivation of $P \xrightarrow{\pi y} P'$.

Solution: According to the definition of $P \mid x$, we have to show that there exists $\bar{z}$, $w$, $P_0$, and $P_1$ such that $P = \nu \pi z. \!(\pi w. P_0 \mid P_1)$. In fact we can always use $w = y$, as we show in the following induction on the derivation of $P \xrightarrow{\pi y} P'$.
Case (lab-out): Then there exists $P_0$ such that $P = \pi y. P_0$. Take $P_1 = 0$ and $\vec{z}$ to be the empty list of names. Then $P \equiv \nu \vec{z}. (\pi y. P_0 \mid P_1)$, as required.

Case (lab-par-l): Then there exists $P_2$ and $Q$ such that $P = P_2 \mid Q$ with the premiss $P_2 \xrightarrow{\pi y}$. Applying the inductive hypothesis to the premiss, there exist $\vec{z}, P_0$, and $P_1$ such that $P_2 \equiv \nu \vec{z}. (\pi y. P_0 \mid P_1)$. Without loss of generality, we may assume that $\vec{z} \cap \text{fn}(Q) = \varnothing$, hence $P = P_2 \mid Q \equiv \nu \vec{z}. (\pi y. P_0 \mid P_1) \mid Q \equiv \nu \vec{z}. (\pi y. P_0 \mid (P_1 \mid Q))$ by scope extrusion, as required.

Case (lab-par-r): Symmetric to the previous case.

Case (lab-new): There exists $P_2$ and $u$ such that $P = \nu u. P_2$ and $u \notin \{x, y\}$, with the premiss $P_2 \xrightarrow{\pi y}$. Applying the inductive hypothesis to the premiss, there exist $\vec{z}, P_0$, and $P_1$ such that $P_2 \equiv \nu \vec{z}. (\pi y. P_0 \mid P_1)$. Hence $P = \nu u. P_2 \equiv \nu u. (\nu \vec{z}. (\pi y. P_0 \mid P_1))$, as required.

(b) Give an example to show that the converse is not true, i.e. find a $P$ such that $P \downarrow x$ but not $P \xrightarrow{\pi y}$.

Solution: Take $P = \nu y. \pi y. Then $P \downarrow x$ but $P$ can only do a bound output on $x$, i.e. $P \xrightarrow{\pi y} 0$. // 3