Concurrency theory

lts, bisimulation and contextual equivalences

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A historical perspective

CSP Hoare defined the semantics of CSP using an axiomatic approach (problem: you cannot execute a program);

CCS Milner defined the operational semantics of CCS in term of a *labelled transition system* and associated *bisimilarity*;

...several attempts to handle mobility algebraically led to...

pi-calculus Milner, Parrow and Walker introduced the pi-calculus. They defined its semantics along the lines of research on CCS, that is, before defining the reduction semantics, they defined an LTS...

Lifting CCS techniques to name-passing is not straightforward

Actually, the original paper on pi-calculus defines *two* LTSs (excerpts):

Early LTS

$$\overline{x}\langle v\rangle.P \xrightarrow{\overline{x}\langle v\rangle} P$$

$$x(y).P \xrightarrow{x(v)} P\{^v/_y\}$$

$$\frac{P \xrightarrow{\overline{x}\langle v\rangle} P' \qquad Q \xrightarrow{x(v)} Q'}{P \mid\mid Q \xrightarrow{\tau} P' \mid\mid Q'}$$

Late LTS

$$\overline{x}\langle v\rangle.P \xrightarrow{\overline{x}\langle v\rangle} P$$

$$x(y).P \xrightarrow{x(y)} P$$

$$\frac{P \xrightarrow{\overline{x}\langle v\rangle} P' \qquad Q \xrightarrow{x(y)} Q'}{P \mid\mid Q \xrightarrow{\tau} P' \mid\mid Q'\{^v/_y\}}$$

These LTSs define the same τ-transitions, where is the problem?

Problem

Definition: Weak bisimilarity, denoted $\approx$, is the largest symmetric relation such that whenever $P \approx Q$ and $P \xrightarrow{\ell} P'$ there exists Q' such that $Q \stackrel{\hat{\ell}}{\Longrightarrow} Q'$ and $P' \approx Q'$.

But the bisimilarity built on top of them observe all the labels: do the resulting bisimilarities coincide? No!

Which is the right one? Which is the role of the LTS?

Back to CCS – reductions

Syntax:

$$P \quad ::= \quad \mathbf{0} \quad | \quad a.P \quad | \quad \overline{a}.P \quad | \quad P \; \big|\big| \; P \quad | \quad (\boldsymbol{\nu} a)P$$

Reduction semantics:

$$a.P \parallel \overline{a}.Q \;\rightarrow\; P \parallel Q \qquad\qquad \frac{P \;\rightarrow\; P'}{(\boldsymbol{\nu} a)P \;\rightarrow\; (\boldsymbol{\nu} a)P'} \qquad\qquad \frac{P \equiv P' \rightarrow Q' \equiv Q}{P \rightarrow Q}$$

where $\equiv$ is defined as:

$$P \parallel \mathbf{0} \equiv P \qquad\qquad P \parallel Q \equiv Q \parallel P \qquad\qquad (P \parallel Q) \parallel R \equiv P \parallel (Q \parallel R)$$

$$(\boldsymbol{\nu} a)P \parallel Q \equiv (\boldsymbol{\nu} a)(P \parallel Q) \text{ if } a \notin \mathrm{fn}(Q)$$

Back to CCS – observational equivalence

Let reduction barbed congruence, denoted $\simeq$, be the largest symmetric relation over processes that is

preserved by contexts: if $P \simeq Q$ then $C[P] \simeq C[Q]$ for all contexts $C[-]$.

barb preserving: if $P \simeq Q$ and $P \downarrow_n$, then $Q \Downarrow_n$.

Remark:

$$P \downarrow n \quad \text{holds if} \quad P \equiv (\boldsymbol{\nu}\tilde{a})(n.P' \mid\mid P'') \text{ or } P \equiv (\boldsymbol{\nu}\tilde{a})(\overline{n}.P' \mid\mid P'') \text{ with } n \notin \{\tilde{a}\}$$

and $P \Downarrow n$ holds if there exists P' such that $P \rightarrow^* P'$ and $P' \downarrow n$.

reduction closed: if $P \simeq Q$ and $P \rightarrow P'$, then there is a Q' such that $Q \rightarrow^* Q'$ and $P' \simeq Q'$ ($\rightarrow^*$ is the reflexive and transitive closure of $\rightarrow$).

The role of bisimilarity

Observation: the definition of bisimilarity does not involve a universal quantification over all contexts!

Question: is there any relationship between (weak) bisimilarity and reduction barbed congruence?

Theorem:

1. $P \approx Q$ implies $P \simeq Q$ (soundness of bisimilarity);

2. $P \simeq Q$ implies $P \approx Q$ (completeness of bisimilarity).

Point 2. does not hold in general.
Point 1. ought to hold (otherwise your LTS/bisimilarity is very odd!).

Soundness and completeness for a fragment of CCS

Consider the fragment of CCS without sums and replication:

$$a.P \xrightarrow{a} P \qquad \overline{a}.P \xrightarrow{\overline{a}} P \qquad \frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\overline{a}} Q'}{P \parallel Q \xrightarrow{\tau} P' \parallel Q'}$$

$$\frac{P \xrightarrow{\ell} P'}{P \parallel Q \xrightarrow{\ell} P' \parallel Q} \qquad \frac{P \xrightarrow{\ell} P' \quad a \notin \mathrm{fn}(\ell)}{(\boldsymbol{\nu} a)P \xrightarrow{\ell} (\boldsymbol{\nu} a)P'}$$

symmetric rules omitted.

Barbs are defined as $P \downarrow a$ iff $P \equiv (\boldsymbol{\nu}\tilde{n})(a.P' \parallel P'')$ or $P \equiv (\boldsymbol{\nu}\tilde{n})(\overline{a}.P' \parallel P'')$ for $a \notin \tilde{n}$.

Soundness of weak bisimilarity: $P \approx Q$ implies $P \simeq Q$.

Proof We show that $\approx$ is contextual, barb preserving, and reduction closed.

Contextuality of $\approx$ can be shown by induction on the structure of the contexts, and by case analysis of the possible interactions between the processes and the contexts. (Congruence of bisimilarity).

Suppose that $P \approx Q$ and $P \downarrow a$. Then $P \equiv (\boldsymbol{\nu}\tilde{n})(a.P_1 \parallel P_2)$, with $a \notin \tilde{n}$. We derive $P \xrightarrow{a} (\boldsymbol{\nu}\tilde{n})(P_1 \parallel P_2)$. Since $P \approx Q$, there exists Q' such that $Q \xRightarrow{a} Q'$, that is $Q \xrightarrow{\tau}{}^{*} Q'' \xrightarrow{a} \ldots$. But Q'' must be of the form $(\boldsymbol{\nu}\tilde{m})(a.Q_1 \parallel Q_2)$ with $a \notin \tilde{m}$. This implies that $Q'' \downarrow a$, and in turn $Q \Downarrow a$, as required.

Suppose that $P \approx Q$ and $P \rightarrowtriangle P'$. We have that $P \xrightarrow{\tau} P'' \equiv P'$. Since $P \approx Q$, there exists Q' such that $Q \xrightarrow{\tau}{}^{*} Q'$ and $P' \equiv P'' \approx Q'$. Since $Q \xrightarrow{\tau}{}^{*} Q'$ it holds that $Q \rightarrowtriangle^{*} Q'$. Since $P' \equiv P''$ implies $P' \approx P''$, by transitivity of $\approx$ we conclude $P' \approx Q'$, as required. $\square$

Completeness of weak bisimilarity: $P \simeq Q$ implies $P \approx Q$.

Proof We show that $\simeq$ is a bisimulation.

Suppose that $P \simeq Q$ and $P \xrightarrow{a} P'$ (the case $P \simeq Q$ and $P \xrightarrow{\tau} P'$ is easy). Let

$$\begin{array}{rcl} C_a[-] & = & - \parallel \overline{a}.d \\ C_{\overline{a}}[-] & = & - \parallel a.d \end{array} \qquad \begin{array}{rcl} Flip & = & \overline{d}.(o \oplus f) \\ -_1 \oplus -_2 & = & (\boldsymbol{\nu} z)(z.\,-_1 \parallel z.\,-_2 \parallel \overline{z}) \end{array}$$

where the names z, o, f, d are *fresh* for P and Q.

Lemma 1. $C_a[P] \rightarrowtriangle^* P' \parallel d$ if and only if $P \stackrel{a}{\Longrightarrow} P'$. Similarly for $C_{\overline{a}}[-]$.

Since $\simeq$ is contextual, we have $C_a[P] \parallel Flip \simeq C_a[Q] \parallel Flip$. By Lemma 1. we have $C_a[P] \parallel Flip \rightarrowtriangle^* P_1 \equiv P' \parallel o \parallel (\boldsymbol{\nu} z)z.f$.

Lemma 2. If $P \simeq Q$ and $P \rightarrowtriangle^* P'$ then there exists Q' such that $Q \rightarrowtriangle^* Q'$ and $P' \simeq Q'$.

By Lemma 2. there exists Q_1 such that $C_a[Q] \;||\; Flip \twoheadrightarrow^* Q_1$ and $P_1 \simeq Q_1$. Now, $P_1 \downarrow o$ and $P_1 \not\downarrow f$. Since $\simeq$ is barb preserving, we have $Q_1 \Downarrow o$ and $Q_1 \not\Downarrow f$. The absence of the barb f implies that the $\oplus$ operator reduced, and in turn that the d action has been consumed: this can only occur if Q realised the a action. Thus we can conclude $Q_1 \equiv Q' \;||\; o \;||\; (\boldsymbol{\nu} z)z.f$, and by Lemma 1. we also have $Q \stackrel{a}{\Longrightarrow} Q'$.

It remains to show that $P' \simeq Q'$.

Lemma 3. $(\boldsymbol{\nu} z)z.P \simeq \mathbf{0}$.

Since $P_1 \simeq Q_1$ and $\simeq$ is contextual, we have $(\boldsymbol{\nu} o)P_1 \simeq (\boldsymbol{\nu} o)Q_1$. By Lemma 3., we have

$$P' \simeq P' \;\Big|\Big|\; (\boldsymbol{\nu} o)o \;\Big|\Big|\; (\boldsymbol{\nu} z)z.f \;\equiv\; (\boldsymbol{\nu} o)P_1 \;\simeq\; (\boldsymbol{\nu} o)Q_1 \;\equiv\; Q' \;\Big|\Big|\; (\boldsymbol{\nu} o)o \;\Big|\Big|\; (\boldsymbol{\nu} z)z.f \simeq Q' \,.$$

The equivalence $P' \simeq Q'$ follows because $\equiv \subseteq \simeq$ and $\simeq$ is transitive. $\square$

Exercise: explain the role of the $Flip$ process.

LTSs revisited

Reduction barbed congruence involves a universal quantification over all contexts. Weak bisimilarity does not, yet bisimilarity *is a sound proof technique* for reduction barbed congruence. How is this possible?

> An LTS captures all the interactions that a term can have with an arbitrary context. In particular, each label correspond to a minimal context.

For instance, in CCS, $P \xrightarrow{a} P'$ denotes the fact that P can interact with the context $C[-] = - \parallel \overline{a}$, yielding P'.

And τ transitions characterises all the interactions with an *empty context*.

Pi-calculus: labels

Given a process P, which are the contexts[1] that yield a reduction?

- if $P \equiv (\boldsymbol{\nu}\tilde{n})(\overline{x}\langle v\rangle.P_1 \mid\mid P_2)$ with $x, v \notin \tilde{n}$, then P interacts with the context

  $$C[-] = - \mid\mid x(y).Q$$

  yielding:

  $$C[P] \rightarrow \underbrace{(\boldsymbol{\nu}\tilde{n})(P_1 \mid\mid P_2)}_{P'} \mid\mid Q\{^v/_y\}$$

  We record this interaction with the label $\overline{x}\langle v\rangle$: $P \xrightarrow{\overline{x}\langle v\rangle} P'$.

[1] to simplify the notations, we will not write the most general contexts.

- if $P \equiv (\boldsymbol{\nu}\tilde{n})(x(y).P_1 \parallel P_2)$ with $x \notin \tilde{n}$, then P interacts with the context

$$C[-] = - \;\Big|\Big|\; \overline{x}\langle v\rangle.Q \qquad \text{for } v \notin \tilde{n}, \text{ yielding:}$$

$$C[P] \rightharpoonup \underbrace{(\boldsymbol{\nu}\tilde{n})(P_1\{^v/_y\} \;\Big|\Big|\; P_2)}_{P'} \;\Big|\Big|\; Q$$

  We record this interaction with the label $x(v)$: $P \xrightarrow{x(v)} P'$

- If $P \rightharpoonup P'$, then P reduces without interacting with a context $C[-] = - \parallel Q$:

$$C[P] \rightharpoonup P' \;\Big|\Big|\; Q$$

  We record this interaction with the label τ: $P \xrightarrow{\tau} P'$.

Intermezzo

What if we define a labelled bisimilarity using the previous labels?

Consider the processes:

$$P = (\boldsymbol{\nu} v)\overline{x}\langle v\rangle \quad \text{and} \quad Q = \mathbf{0}$$

Obviously, $P \not\simeq Q$ because $P \downarrow x$ while $Q \not\Downarrow x$.

But both P and Q realise no labels: they are equated by the bisimilarity.

The bisimilarity is not *sound*!

Maybe we forgot a label...

The missing interaction

- if $P \equiv (\boldsymbol{\nu}\tilde{n})(\overline{x}\langle v\rangle.P_1 \parallel P_2)$ with $x \notin \tilde{n}$ and $v \in \tilde{n}$, then P interacts with the context

  $$C[-] = - \,\big|\big|\, x(y).Q$$

  yielding:

  $$C[P] \rightarrow (\boldsymbol{\nu}v)(\underbrace{(\boldsymbol{\nu}\tilde{n} \setminus v)(P_1 \,\big|\big|\, P_2)}_{P'} \,\big|\big|\, Q\{^v/_y\})$$

  We record this interaction with the label $(\boldsymbol{\nu}v)\overline{x}\langle v\rangle$: $P \xrightarrow{(\boldsymbol{\nu}v)\overline{x}\langle v\rangle} P'$.

  Intuition: in P' the scope of v has been opened.

Summary of actions

ℓ	kind	$\mathrm{fn}(\ell)$	$\mathrm{bn}(\ell)$	$\mathsf{n}(\ell)$
$\overline{x}\langle y\rangle$	free output	$\{x, y\}$	$\emptyset$	$\{x, y\}$
$(\boldsymbol{\nu} y)\overline{x}\langle y\rangle$	bound output	$\{x\}$	$\{y\}$	$\{x, y\}$
$x(y)$	input	$\{x, y\}$	$\emptyset$	$\{x, y\}$
τ	internal	$\emptyset$	$\emptyset$	$\emptyset$

Pi-calculus: LTS

$$\overline{x}\langle v\rangle.P \xrightarrow{\overline{x}\langle v\rangle} P \qquad x(y).P \xrightarrow{x(v)} P\{{}^{v}/_{y}\} \qquad \frac{P \xrightarrow{\overline{x}\langle v\rangle} P' \qquad Q \xrightarrow{x(v)} Q'}{P \parallel Q \xrightarrow{\tau} P' \parallel Q'}$$

$$\frac{P \xrightarrow{\ell} P' \qquad \mathrm{bn}(\ell) \cap \mathrm{fn}(Q) = \emptyset}{P \parallel Q \xrightarrow{\ell} P' \parallel Q} \qquad \frac{P \xrightarrow{\ell} P' \qquad v \notin \mathsf{n}(\ell)}{(\boldsymbol{\nu} v)P \xrightarrow{\ell} (\boldsymbol{\nu} v)P'} \qquad \frac{P \parallel !P \xrightarrow{\ell} P'}{!P \xrightarrow{\ell} P'}$$

$$\frac{P \xrightarrow{\overline{x}\langle v\rangle} P' \qquad x \neq v}{(\boldsymbol{\nu} v)P \xrightarrow{(\boldsymbol{\nu} v)\overline{x}\langle v\rangle} P'} \qquad \frac{P \xrightarrow{(\boldsymbol{\nu} v)\overline{x}\langle v\rangle} P' \qquad Q \xrightarrow{x(v)} Q' \qquad v \notin \mathrm{fn}(Q)}{P \parallel Q \xrightarrow{\tau} (\boldsymbol{\nu} v)(P' \parallel Q')}$$

Pi-calculus: bisimilarity

We can define bisimilarity for pi-calculus in the standard way.

Let $\stackrel{\hat{\ell}}{\Longrightarrow}$ be $\stackrel{\tau}{\longrightarrow}^* \stackrel{\ell}{\longrightarrow} \stackrel{\tau}{\longrightarrow}^*$ if $\ell \neq \tau$, and $\stackrel{\tau}{\longrightarrow}^*$ otherwise.

Definition: Weak bisimilarity, denoted $\approx$, is the largest symmetric relation such that whenever $P \approx Q$ and $P \stackrel{\ell}{\longrightarrow} P'$ there exists Q' such that $Q \stackrel{\hat{\ell}}{\Longrightarrow} Q'$ and $P' \approx Q'$.

Last week examples

1. $\overline{x}\langle y\rangle \not\approx \mathbf{0}$: trivial because $\overline{x}\langle y\rangle \xrightarrow{\overline{x}\langle y\rangle}$ and $\mathbf{0} \not\xrightarrow{\overline{x}\langle y\rangle}$.

2. $(\boldsymbol{\nu}x)\overline{x}\langle\rangle.R \approx \mathbf{0}$: the relation $\mathcal{R} = \{((\boldsymbol{\nu}x)\overline{x}\langle\rangle.R, \mathbf{0})\}^{=}$ is a bisimulation.

3. $(\boldsymbol{\nu}x)(\overline{x}\langle y\rangle.R_1 \mid\mid x(z).R_2) \approx (\boldsymbol{\nu}x)(R_1 \mid\mid R_2\{^y/_z\})$

   The relation

$$\mathcal{R} = \{((\boldsymbol{\nu}x)(\overline{x}\langle y\rangle.R_1 \;\Big|\Big|\; x(z).R_2), (\boldsymbol{\nu}x)(R_1 \;\Big|\Big|\; R_2\{^y/_z\}))\}^{=} \;\cup\; \mathcal{I}$$

   is a bisimulation.

   $\mathcal{I}$ is the identity relation over processes, and $\mathcal{R}^{=}$ denotes the symmetric closure of $\mathcal{R}$.

Subtleties of pi-calculus LTS

Exercise: derive a τ transition corresponding to this reduction:

$$(\boldsymbol{\nu} x)\overline{a}\langle x\rangle.P \;\|\; a(y).Q \;\rightarrow\; (\boldsymbol{\nu} x)(P \;\|\; Q\{^x/_y\})$$

Exercise: each side condition in the definition of the LTS is needed to have the theorem

$$P \rightarrow Q \text{ iff } P \xrightarrow{\tau}\equiv Q$$

Remove on side condition at a time and find counter-examples to this theorem.

Weak bisimulation is a sound proof technique for reduction barbed congruence

- Prove that weak bisimulation is *reduction closed*.

  ...at the blackboard

- Prove that weak bisimulation is *barb preserving*.

  ...at the blackboard

- Prove that weak-bisimulation is a congruence.

  ...ahem, think twice...

On soundness of weak bisimilarity

Exercise: Consider the terms (in a pi-calculus extended with $+$):

$$P = \overline{x}\langle v\rangle \;||\; y(z)$$

$$Q = \overline{x}\langle v\rangle.y(z) + y(z).\overline{x}\langle v\rangle$$

1. Prove that $P \approx Q$[2].

2. Does $P \simeq Q$?[3]

[2] Does this hold if we replace $+$ by $-_1 \oplus -_2 = (\boldsymbol{\nu} w)(\overline{w}\langle\rangle \;||\; w().-_1 \;||\; w().-_2)$ in Q?

[3] Hint: define a context that *equates* the names x and y.

Bisimilarity is not a congruence

In pi-calculus, bisimilarity (both strong and weak) is not preserved by input prefixes, that is contexts of the form $C[-] = x(y).-$. *When we built the labels, we forgot the contexts which can interact with the process by changing its internal structure.*

Question: how to recover the soundness of the bisimilarity with respect to the reduction barbed congruence? Two solutions:

1. close the reduction barbed congruence under *all non input prefix contexts*;

2. close the bisimilarity under substitution: let $P \approx^c Q$ (P is *fully bisimilar* with Q) if $P\sigma \approx Q\sigma$ for all substitutions σ.

Exercise: Show that $P \not\approx^c Q$, where P and Q are defined in the previous slide.

And completeness?

Completeness of bisimulation with respect to barbed congruence[4] (closed under non-input prefixes, denoted $\simeq^-$) holds in the strong case. In the weak case, we have that for

$$P = \overline{a}\langle x \rangle \parallel E_{xy} \qquad Q = \overline{a}\langle y \rangle \parallel E_{xy}$$

where

$$E_{xy} = !x(z).\overline{y}\langle z \rangle \parallel !y(z).\overline{x}\langle z \rangle$$

it holds that $P \not\approx Q$ but $P \simeq^- Q$ for each context $C[-]$.

Completeness (for image-finite processes) holds if a name-matching operator is added to the language.

[4]barbed congruence is a variant of reduction-closed barbed congruence in which closure under context is allowed only at the beginning of the bisimulation game.

Asynchronous communication

CCS and pi-calculus (and many others) are based on *synchronized interaction*, that is, the acts of sending a datum and receiving it coincide:

$$\overline{a}.P \mid\mid a.Q \rightarrow P \mid\mid Q \ .$$

In real-world distributed systems, sending a datum and receiving it are *distinct acts*:

$$\overline{a}.P \mid\mid a.Q \ \ldots \rightarrow \ldots \ \overline{a} \mid\mid P \mid\mid a.Q \ \ldots \rightarrow \ldots \ P' \mid\mid Q \ .$$

In an *asynchronous* world, the prefix . does not express temporal precedence.

Asynchronous interaction made easy

Idea: the only term than can appear underneath an output prefix is $\mathbf{0}$.

Intuition: an unguarded occurence of $\overline{x}\langle y\rangle$ can be thought of as a datum y in an implicit communication medium tagged with x.

Formally:

$$\overline{x}\langle y\rangle \;\Big|\Big|\; x(z).P \;\rightarrow\; P\{^{y}\!/_{z}\}\ .$$

We suppose that the communication medium has unbounded capacity and preserves no ordering among output particles.

Asynchronous pi-calculus

Syntax:

$$P \ ::= \ \mathbf{0} \ \mid \ x(y).P \ \mid \ \overline{x}\langle y\rangle \ \mid \ P \,||\, P \ \mid \ (\boldsymbol{\nu} x)P \ \mid \ !P$$

The definitions of free and bound names, of structural congruence $\equiv$, and of the reduction relation $\twoheadrightarrow$ are inherited from pi-calculus.

Examples

Sequentialization of output actions is still possible:

$$(\boldsymbol{\nu} y, z)(\overline{x}\langle y\rangle \;||\; \overline{y}\langle z\rangle \;||\; \overline{z}\langle a\rangle \;||\; R)\ .$$

Synchronous communication can be implemented by waiting for an acknoledgement:

$$\begin{aligned} [\![\, \overline{x}\langle y\rangle.P \,]\!] &= (\boldsymbol{\nu} u)(\overline{x}\langle y, u\rangle \;||\; u().P) \\ [\![\, x(v).Q \,]\!] &= x(v, w).(\overline{w}\langle\rangle \;||\; Q) \qquad \text{for } w \notin Q \end{aligned}$$

Exercise: implement synchronous communication without relying on polyadic primitives.

Contextual equivalence and asynchronous pi-calculus

It is natural to impose two constraints to the basic recipe:

- compare terms using only *asynchronous contexts*;
- restrict the observables to be *co-names*. To observe a process *is* to interact with it by performing a complementary action and reporting it: in asynchronous pi-calculus *input actions cannot be observed*.

A peculiarity of synchronous equivalences

The terms

$$P = !x(z).\overline{x}\langle z\rangle$$
$$Q = \mathbf{0}$$

are not reduction barbed congruent, but they are asynchronous reduction barbed congruent.

Intuition: in an asynchronous world, if the medium is unbound, then buffers do not influence the computation.

A proof method

Consider now the weak bisimilarity $\approx_s$ built on top of the standard (early) LTS for pi-calculus. As asynchronous pi-calculus is a sub-calculus of pi-calculus, $\approx_s$ is an equivalence for asynchronous pi-calculus terms.

It holds $\approx_s \subseteq \simeq$, that is the standard pi-calculus bisimilarity is a *sound proof technique for* $\simeq$.

But

$$!x(z).\overline{x}\langle z\rangle \;\; \not\approx_s \mathbf{0} \;.$$

Question: can a labelled bisimilarity recover the natural contextual equivalence?

A problem and two solutions

Transitions in an LTS should represent observable interactions a term can engage with a context:

- if $P \xrightarrow{\overline{x}\langle y \rangle} P'$ then P can interact with the context $- \parallel x(u).\mathrm{beep}$, where beep is activated if and only if the output action has been observed;
- if $P \xrightarrow{x(y)} P'$ then in no way beep can be activated if and only if the input action has been observed!

Solutions:

1. relax the matching condition for input actions in the bisimulation game;
2. modify the LTS so that it precisely identifies the interactions that a term can have with its environment.

Amadio, Castellani, Sangiorgi - 1996

Idea: relax the matching condition for input actions.

Let *asynchronous bisimulation* $\approx_a$ be the largest symmetric relation such that whenever $P \approx_a Q$ it holds:

1. if $P \xrightarrow{\ell} P'$ and $\ell \neq x(y)$ then there exists Q' such that $Q \stackrel{\hat{\ell}}{\Longrightarrow} Q'$ and $P' \approx_a Q'$;

2. if $P \xrightarrow{x(y)} P'$ then there exists Q' such that $Q \mid\mid \overline{x}\langle y \rangle \Longrightarrow Q'$ and $P' \approx_a Q'$.

Remark: P' is the outcome of the interaction of P with the context $- \mid\mid \overline{x}\langle y \rangle$. Clause 2. allows Q to interact with the same context, but does not force this interaction.

Honda, Tokoro - 1992

$$\overline{x}\langle y\rangle \xrightarrow{\overline{x}\langle y\rangle} \mathbf{0} \qquad x(u).P \xrightarrow{x(y)} P\{^y/_u\} \qquad \mathbf{0} \xrightarrow{x(y)} \overline{x}\langle y\rangle$$

$$\frac{P \xrightarrow{\overline{x}\langle y\rangle} P' \quad x \neq y}{(\boldsymbol{\nu} y)P \xrightarrow{(\boldsymbol{\nu} y)\overline{x}\langle y\rangle} P'} \qquad \frac{P \xrightarrow{\alpha} P' \quad y \notin \alpha}{(\boldsymbol{\nu} y)P \xrightarrow{\alpha} (\boldsymbol{\nu} y)P'}$$

$$\frac{P \xrightarrow{\overline{x}\langle y\rangle} P' \quad Q \xrightarrow{x(y)} Q'}{P \parallel Q \xrightarrow{\tau} P' \parallel Q'} \qquad \frac{P \xrightarrow{(\boldsymbol{\nu} y)\overline{x}\langle y\rangle} P' \quad Q \xrightarrow{x(y)} Q' \quad y \notin \mathrm{fn}(Q)}{P \parallel Q \xrightarrow{\tau} (\boldsymbol{\nu} y)(P' \parallel Q')}$$

$$\frac{P \xrightarrow{\alpha} P' \quad \mathrm{bn}(\alpha) \cap \mathrm{fn}(Q) = \emptyset}{P \parallel Q \xrightarrow{\alpha} P' \parallel Q} \qquad \frac{P \equiv P' \quad P' \xrightarrow{\alpha} Q' \quad Q' \equiv Q}{P \xrightarrow{\alpha} Q}$$

Honda, Tokoro explained

Ideas:

- modify the LTS so that it precisely identifies the interactions that a term can have with its environment;
- rely on a standard weak bisimulation.

Amazing results: asynchrounous bisimilarity in ACS style, bisimilarity on top of HT LTS, and barbed congruence coincide.[5]

[5]ahem, modulo some technical details.

Properties of asynchronous bisimilarity in ACS style

- Bisimilarity is a congruence;

  it is preserved also by input prefix, while it is not in the synchronous case;

- bisimilarity is an equivalence relation (transitivity is non-trivial);

- bisimilarity is *sound* with respect to reduction barbed congruence;

- bisimilarity is *complete* with respect to barbed congruence.[6]

[6] for completeness the calculus must be equipped with a matching operator.

Some proofs about ACS bisimilarity... on asynchronous CCS

Syntax:

$$P \;::=\; \mathbf{0} \;\;|\;\; a.P \;\;|\;\; \overline{a} \;\;|\;\; P \,\|\, P \;\;|\;\; (\boldsymbol{\nu} a)P\ .$$

Reduction semantics:

$$a.P \parallel \overline{a} \;\rightarrowtriangle\; P \qquad\qquad \frac{P \equiv P' \rightarrowtriangle Q' \equiv Q}{P \rightarrowtriangle Q}$$

where $\equiv$ is defined as:

$$P \parallel Q \equiv Q \parallel P \qquad\qquad (P \parallel Q) \parallel R \equiv P \parallel (Q \parallel R)$$

$$(\boldsymbol{\nu} a)P \parallel Q \equiv (\boldsymbol{\nu} a)(P \parallel Q) \text{ if } a \notin \mathrm{fn}(Q)$$

Background: LTS and weak bisimilarity for asynchronous CCS

$$a.P \xrightarrow{a} P \qquad \overline{a} \xrightarrow{\overline{a}} \mathbf{0} \qquad \frac{P \xrightarrow{a} P' \quad Q \xrightarrow{\overline{a}} Q'}{P \parallel Q \xrightarrow{\tau} P' \parallel Q'}$$

$$\frac{P \xrightarrow{\ell} P'}{P \parallel Q \xrightarrow{\ell} P' \parallel Q} \qquad \frac{P \xrightarrow{\ell} P' \quad a \notin \mathrm{fn}(\ell)}{(\boldsymbol{\nu} a)P \xrightarrow{\ell} (\boldsymbol{\nu} a)P'}$$

symmetric rules omitted.

Definition: Asynchronous weak bisimilarity, denoted $\approx$, is the largest symmetric relation such that whenever $P \approx Q$ and

- $P \xrightarrow{\ell} P'$, $\ell \in \{\tau, \overline{a}\}$, there exists Q' such that $Q \stackrel{\hat{\ell}}{\Longrightarrow} Q'$ and $P' \approx Q'$;
- $P \xrightarrow{a} P'$, there exists Q' such that $Q \parallel \overline{a} \Longrightarrow Q'$ and $P' \approx Q'$.

Sketch of the proof of transitivity of $\approx$

Let $\mathcal{R} = \{(P, R) : P \approx Q \approx R\}$. We show that $\mathcal{R} \subseteq \approx$.

- Suppose that $P \; \mathcal{R} \; R$ because $P \approx Q \approx R$, and that $P \xrightarrow{a} P'$.

The definition of $\approx$ ensures that there exists Q' such that $Q \; || \; \overline{a} \Longrightarrow Q'$ and $P' \approx Q'$.

Since $\approx$ is a congruence and $Q \approx R$, it holds that $Q \; || \; \overline{a} \approx R \; || \; \overline{a}$.

A simple corollary of the defintion of the bisimilarity ensures that there exists R' such that $R \; || \; \overline{a} \Longrightarrow R'$ and $Q' \approx R'$.

Then $P' \; \mathcal{R} \; R'$ by construction of $\mathcal{R}$.

- The other cases are standard.

Remark the unusual use of the congruence of the bisimilarity.

Sketch of the proof of completeness

We show that $\simeq\ \subseteq\ \approx$.

- Suppose that $P \simeq Q$ and that $P \xrightarrow{a} P'$.

We must conclude that there exists Q' such that $Q \parallel \overline{a} \Longrightarrow Q'$ and $P' \simeq Q'$.

Since $\simeq$ is a congruence, it holds that $P \parallel \overline{a} \simeq Q \parallel \overline{a}$.

Since $P \xrightarrow{a} P'$, it holds that $P \parallel \overline{a} \xrightarrow{\tau} P'$.

Since $P \parallel \overline{a} \simeq Q \parallel \overline{a}$, the definition of $\simeq$ ensures that there exists Q' such that $Q \parallel \overline{a} \Longrightarrow Q'$ and $P' \simeq Q'$, as desired.

- The other cases are analogous to the completeness proof in synchronous CCS.

The difficulty of the completeness proof is to construct contexts that observe the actions of a process. The case $P \xrightarrow{a} P'$ is straightforward because "there is nothing to observe".

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