SN and redex creation in higher-order typed λ-calculus

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Plan

- Higher-order typed λ-calculus
- Weak vs Strong normalization
- Redex creation and strong normalization
- Girard's proof for strong normalization
- Finite developments
- Open problem

1st&2nd-order typing rules

(variable) $$\frac{}{\Gamma, x{:}\tau \vdash x{:}\tau}$$

(application) $$\frac{\Gamma \vdash M : \sigma \to \tau \quad \Gamma \vdash N{:}\sigma}{\Gamma \vdash MN : \tau}$$

(abstraction) $$\frac{\Gamma, x{:}\sigma \vdash M{:}\tau}{\Gamma \vdash \lambda x.M : \sigma \to \tau}$$

(1st-order typing)

$$\frac{\Gamma \vdash M : \forall\alpha.\tau}{\Gamma \vdash M : \tau\{\alpha := \sigma\}}$$

$$\frac{\Gamma \vdash M{:}\tau \quad \alpha \notin \mathrm{TVar}(\Gamma)}{\Gamma \vdash M : \forall\alpha.\tau}$$

higher-order typing rules

(axioms) $<>\vdash c : s,$ if $(c : s) \in \mathcal{A}$;

(start) $$\frac{\Gamma \vdash A : s}{\Gamma, x : A \vdash x : A},$$ if $x \equiv {}^s x \notin \Gamma$;

(weakening) $$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s}{\Gamma, x : C \vdash A : B},$$ if $x \equiv {}^s x \notin \Gamma$;

(product) $$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x{:}A \vdash B : s_2}{\Gamma \vdash (\Pi x{:}A.B) : s_3},$$ if $(s_1, s_2, s_3) \in \mathcal{R}$;

(application) $$\frac{\Gamma \vdash F : (\Pi x{:}A.B) \quad \Gamma \vdash a : A}{\Gamma \vdash Fa : B[x := a]};$$

(abstraction) $$\frac{\Gamma, x{:}A \vdash b : B \quad \Gamma \vdash (\Pi x{:}A.B) : s}{\Gamma \vdash (\lambda x{:}A.b) : (\Pi x{:}A.B)};$$

(conversion) $$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s \quad B =_\beta B'}{\Gamma \vdash A : B'}.$$

Usual sorts

$(s_1, s_2, s_3) = (s_1, s_2, s_2)$

where (s_1, s_2) possible values are:

$\lambda\!\to$	$(*,*)$			
$\lambda 2$	$(*,*)$	$(\Box,*)$		
λP	$(*,*)$		$(*,\Box)$	
$\lambda P2$	$(*,*)$	$(\Box,*)$	$(*,\Box)$	
$\lambda\underline{\omega}$	$(*,*)$			$(\Box,\Box)$
$\lambda\omega$	$(*,*)$	$(\Box,*)$		$(\Box,\Box)$
$\lambda P\underline{\omega}$	$(*,*)$		$(*,\Box)$	$(\Box,\Box)$
$\lambda P\omega = \lambda C$	$(*,*)$	$(\Box,*)$	$(*,\Box)$	$(\Box,\Box)$

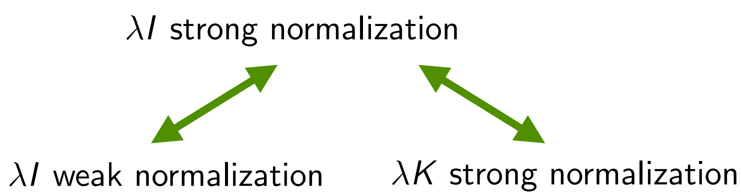
Usual abbrevs

$$\forall\alpha.A \equiv \Pi\alpha{:}*.A$$

$$\Lambda\alpha.M \equiv \lambda\alpha{:}*.M$$

$A \to B \equiv \Pi\, x : A\,.\, B$ when $x \notin$ FVar(B)

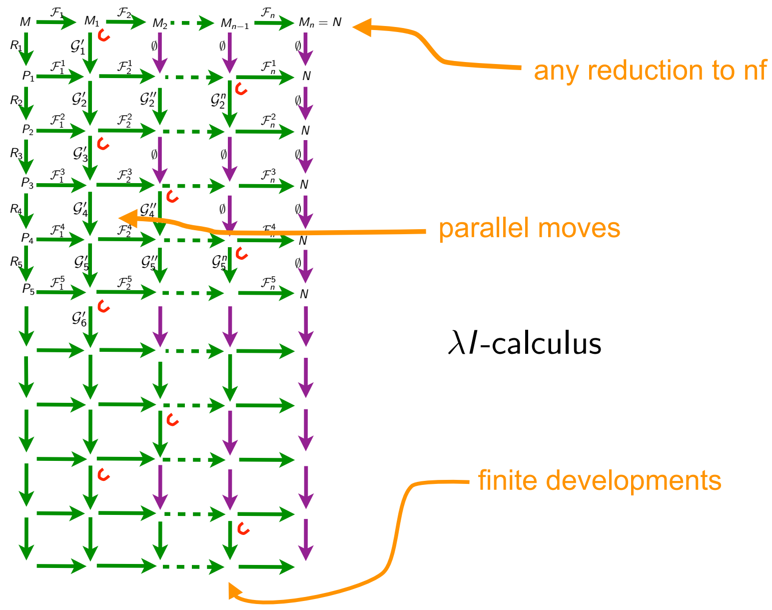
Weak vs Strong Normalisation



- true in any PTS lambda system

[conjecture Barendregt / Geuvers]

Weak vs Strong Normalisation



Weak normalization in lambda-I

- innermost reduction clearly terminates (in lambda-K fst order)

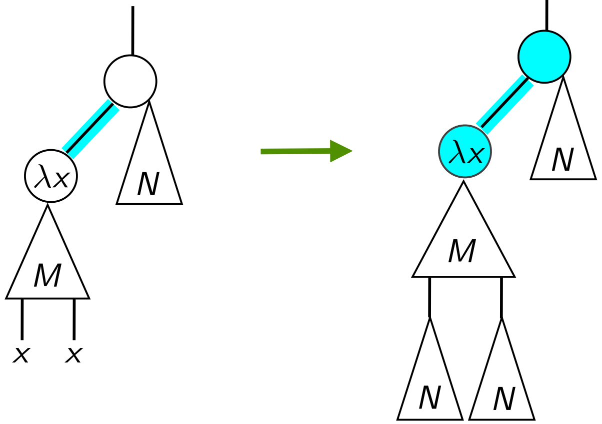
(take multiset ordering on degrees of redexes)

- weak implies strong in lambda-I

(take same argument as for standardization proof: finite developments + cube lemma)

Weak vs Strong Normalisation

- Nederpelt[72], Klop[80], Sorensen[?]



Strong Normalisation (1st order)

- why typed 1st-order calculus normalizes ?

$\underbrace{(\lambda x.\cdots xN\cdots)}_{\sigma\to\tau}\underbrace{(\lambda y.M)}_{\sigma} \rightarrow \cdots\underbrace{(\lambda y.M)N'}_{\sigma}\cdots$ — creates

creation downward

creation upward

$\underbrace{(\lambda x.\underbrace{\lambda y.M}_{\tau})}_{\sigma\to\tau}NP \rightarrow \underbrace{(\lambda y.M')}_{\tau}P$ — creates

$\underbrace{(\lambda x.x)}_{\tau\to\tau}\underbrace{(\lambda y.M)}_{\tau}N \rightarrow \underbrace{(\lambda y.M)}_{\tau}N$ — creates

- degree of a redex is type of its function part
- degree strictly decreases with creation

Strong Normalisation(2nd order)

- why system F normalizes ?

$\underbrace{(\lambda x.\cdots xx\cdots)}_{\tau\to\tau}\underbrace{(\lambda y.y)}_{\tau} \rightarrow \cdots\underbrace{(\lambda y.y)}_{\tau\to\tau}(\lambda y.y)\cdots$ — creates

where $\tau = \forall\alpha\,.\,\alpha\to\alpha$

$\underbrace{(\lambda x.\underbrace{\lambda y.M}_{\tau})}_{\sigma\to\tau}NP \rightarrow \underbrace{(\lambda y.M')}_{\tau}P$ — creates

Strong Normalisation(2nd order)

- looking more closely at system F

$\underbrace{(\lambda x.\cdots xx\cdots)}_{\tau\to\tau}\underbrace{(\lambda y.y)}_{\tau} \rightarrow \cdots\underbrace{(\lambda y.y)}_{\tau\to\tau}(\lambda y.y)\cdots$ — creates

where $\tau = \forall\alpha\,.\,\alpha\to\alpha$

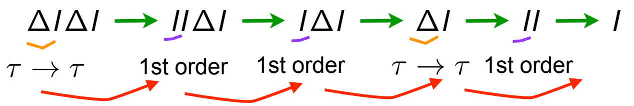
$(\lambda x.\cdots xx\cdots)(\lambda y.y) \rightarrow \cdots(\lambda y.y)(\lambda y.y)\cdots$

2nd order

fst order !

also typable with $\forall\alpha\,.\,\tau'\to\tau'$ where $\tau' = \alpha\to\alpha$

Strong Normalisation(2nd order)



where

$\tau = \forall\alpha\,.\,\alpha \to \alpha$

$\Delta = \lambda x.xx$

$I = \lambda x.x$

Girard - Tait - Krivine proof

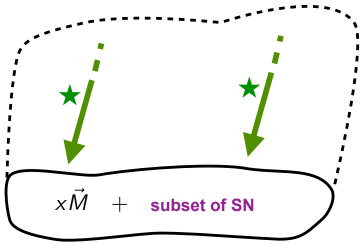
- **Definition** (**saturated sets**) $X \in \mathrm{SAT}$ iff $X \subset \mathcal{SN}$ and

(1) $x\vec{M} \in X$ when $\vec{M} \in \mathcal{SN}$

(2) $M\{x := N\}\vec{P} \in X$ and $N \in \mathcal{SN}$ implies $(\lambda x.M)N\vec{P} \in X$

(1) = non emptyness **(2) = closed by SN-head-beta-expansion**

A saturated set



Girard - Tait - Krivine proof

Let $\mathcal{N}_0 = \{x\vec{M} \mid \vec{M} \in \mathcal{S}\vec{\mathcal{N}}\}$

and $X \to Y = \{M \mid N \in X \Rightarrow MN \in Y\}$

- **Fact 1** $\mathcal{N}_0 \subset \mathcal{SN} \to \mathcal{N}_0 \subset \mathcal{N}_0 \to \mathcal{SN} \subset \mathcal{SN}$
- **Fact 2** $\mathcal{SN} \in \mathrm{SAT}$

- **Lemma 1** $X, Y \in \mathrm{SAT}$ implies $X \to Y \in \mathrm{SAT}$
- **Lemma 2** $X_i \in \mathrm{SAT}$ implies $\bigcap_{i \in I} X_i \in \mathrm{SAT}$

Girard - Tait - Krivine proof

- **Semantics of types** Let $\zeta \in \mathrm{TVar} \to \mathrm{SAT}$. Then $[\![\tau]\!]_\zeta$ is

$[\![\alpha]\!]_\zeta = \zeta(\alpha)$

$[\![\sigma \to \tau]\!]_\zeta = [\![\sigma]\!]_\zeta \to [\![\tau]\!]_\zeta \qquad [\![\forall\alpha.\tau]\!]_\zeta = \bigcap_{X \in \mathrm{SAT}} [\![\tau]\!]_{\zeta\{\alpha \mapsto X\}}$

- **Corollary (1-2)** $[\![\tau]\!]_\zeta \in \mathrm{SAT}$
- **Lemma 3** (**subst**) $[\![\tau\{\alpha := \sigma\}]\!]_\zeta = [\![\tau]\!]_{\zeta\{\alpha \mapsto [\![\sigma]\!]_\zeta\}}$
- **Lemma 4** Let $x_1{:}\tau_1, \dots x_n{:}\tau_n \vdash M{:}\tau$ and $N_1 \in [\![\tau_1]\!]_\zeta, \dots N_n \in [\![\tau_n]\!]_\zeta$
  Then $M\{x_1 := N_1, \dots x_n := N_n\} \in [\![\tau]\!]_\zeta$
- **Corollary (4)** $\Gamma \vdash M{:}\tau$ implies $M \in \mathcal{SN}$

Girard - Tait - Krivine proof

- **Semantics of terms** Let $\rho \in \mathrm{Var} \to \Lambda$. Then

  $[\![M]\!]_\rho = M\{x_1 := \rho(x_1), \dots x_n := \rho(x_n)\}$

  $\rho, \zeta \models M{:}\tau$ iff $[\![M]\!]_\rho \in [\![\tau]\!]_\zeta$

  $\rho, \zeta \models \Gamma$ iff $\rho, \zeta \models x{:}\tau$ for any $(x{:}\tau) \in \Gamma$

  $\Gamma \models M{:}\tau$ iff $\forall \rho, \zeta \quad \rho, \zeta \models \Gamma \Rightarrow \rho, \zeta \models M{:}\tau$

- **Lemma 3** (**subst**) $[\![\tau\{\alpha := \sigma\}]\!]_\zeta = [\![\tau]\!]_{\zeta\{\alpha \mapsto [\![\sigma]\!]_\zeta\}}$
- **Lemma 4** $\Gamma \vdash M{:}\tau$ implies $\Gamma \models M{:}\tau$
- **Corollary** $\Gamma \vdash M{:}\tau$ implies $M \in \mathcal{SN}$

Simple higher-order calculus

$$M, N, A, B, \dots ::= x \mid MN \mid \lambda x{:}A\,.\,M \mid \Pi x{:}A\,.\,B$$

$$(\lambda x{:}A\,.\,M)N \longrightarrow M\{x := N\}$$

The 2 theorems

- Confluence
- Strong normalisation in typed calculi when sorts are well-founded

Tracking redexes in untyped calculus

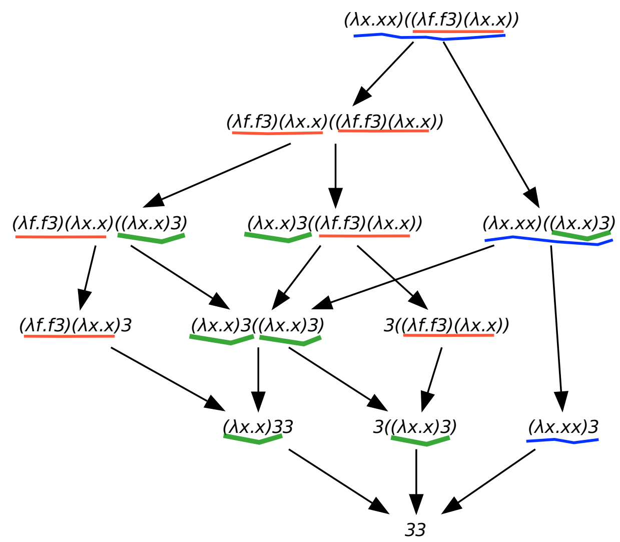
$$M, N, \dots ::= x \mid MN \mid \lambda x\,.\,M$$

$$(\lambda x\,.\,M)N \longrightarrow M\{x := N\}$$

The 2 theorems

- Confluence
- Finite developments (cube lemma)

Redex families



- 3 redex families: **red**, **blue**, **green**.

Tracking redexes in untyped calculus

$$M, N, \ldots ::= {}^{\alpha}x \mid {}^{\alpha}(MN) \mid {}^{\alpha}(\lambda x\,.\,M)$$

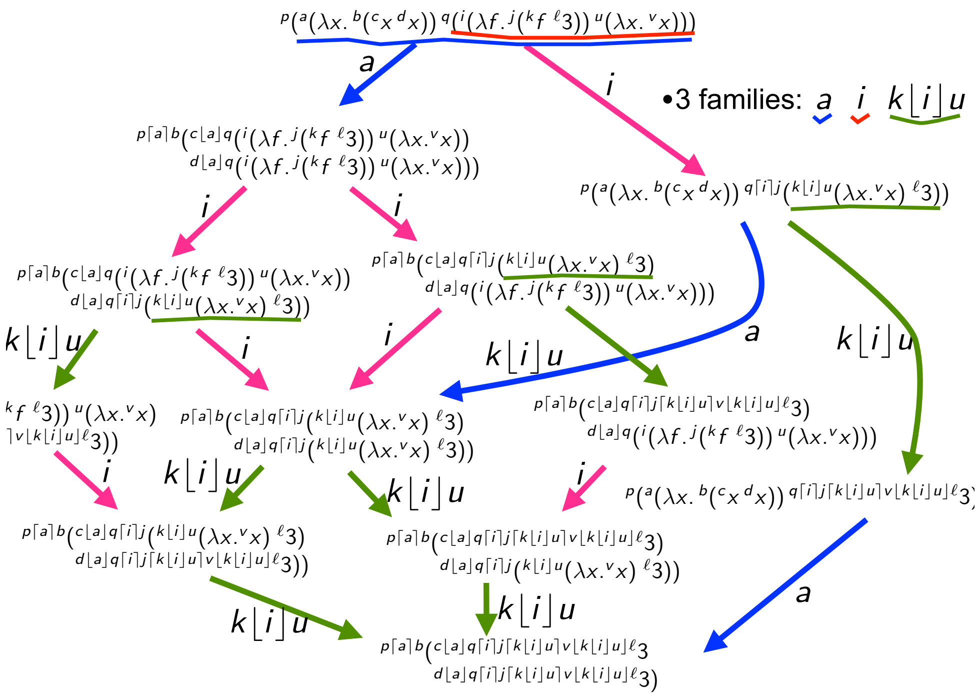
$$^{\beta}({}^{\alpha}(\lambda x\,.\,M)N) \longrightarrow {}^{\beta\lceil\alpha\rceil}M\{x := {}^{\lfloor\alpha\rfloor}N\}$$

where

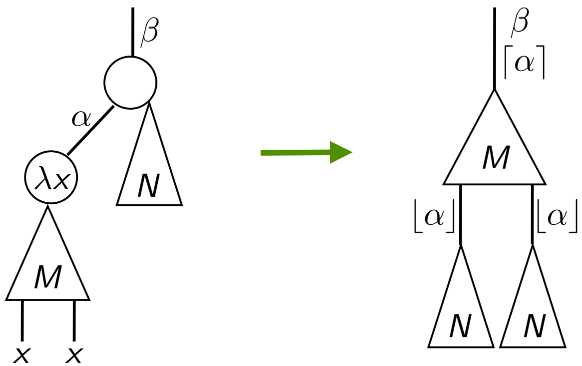
$$^{\alpha}({}^{\beta}U) = {}^{\alpha\beta}U \quad \text{and} \quad {}^{\alpha}x\{x := M\} = {}^{\alpha}M$$

The 2 theorems

- Confluence (consistent names of redexes)
- Created redexes contain names of creators



Graphically



Finite and infinite reductions (1/3)

- **Definition** A **reduction relative** to a set $\mathcal{F}$ of redex families is any reduction contracting redexes in families of $\mathcal{F}$.

  A **development** of $\mathcal{F}$ is any maximal relative reduction.

- **Theorem** `[Finite Developments+, 76]`
  Let $\mathcal{F}$ be a finite set of redex families.

  (1) there are no infinite reductions relative to $\mathcal{F}$,
  (2) they all finish on same term N
  (3) All developments are equivalent by permutations.

Finite and infinite reductions (2/3)

- **Corollary** An **infinite reduction** contracts an **infinite set of redex families.**

- **Corollary** The first-order typed λ-calculus strongly terminates.

Proof In first-order typed λ-calculus:

(1) residuals $R' = (\lambda x.M')N'$ of $R = (\lambda x.M)N$ keep the degree

(2) new redexes have lower degree

Tracking redexes in HO calculus

$$M, N, A, B, \ldots ::= {}^{\alpha}x \mid {}^{\alpha}(MN) \mid {}^{\alpha}(\lambda x{:}A\,.\,M) \mid {}^{\alpha}(\Pi x{:}A\,.\,B)$$

$${}^{\beta}({}^{\alpha}(\lambda x{:}A\,.\,M)N) \longrightarrow {}^{\beta\lceil\alpha,A\rceil}M\{x := {}^{\lfloor\alpha,A\rfloor}N\}$$

where

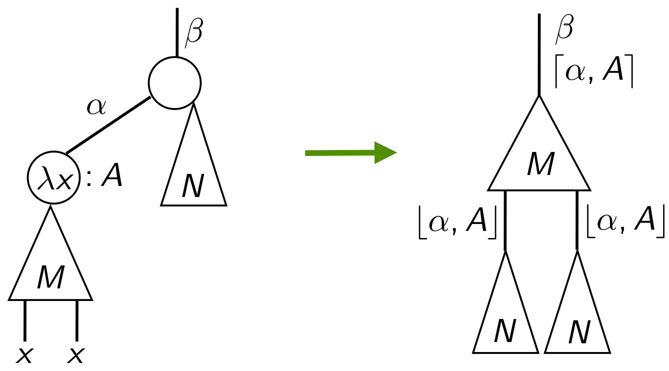
$${}^{\alpha}({}^{\beta}U) = {}^{\alpha\beta}U \quad \text{and} \quad {}^{\alpha}x\{x := M\} = {}^{\alpha}M$$

and $({}^{\lceil\alpha,A\rceil}M)\{x := N\} = {}^{\lceil\alpha,A\{x:=N\}\rceil}M\{x := N\}$

The 1 theorem

- Confluence

Graphically



Example

$$\Delta I \longrightarrow II \longrightarrow I \qquad \tau = \forall t\,.\,t \to t$$

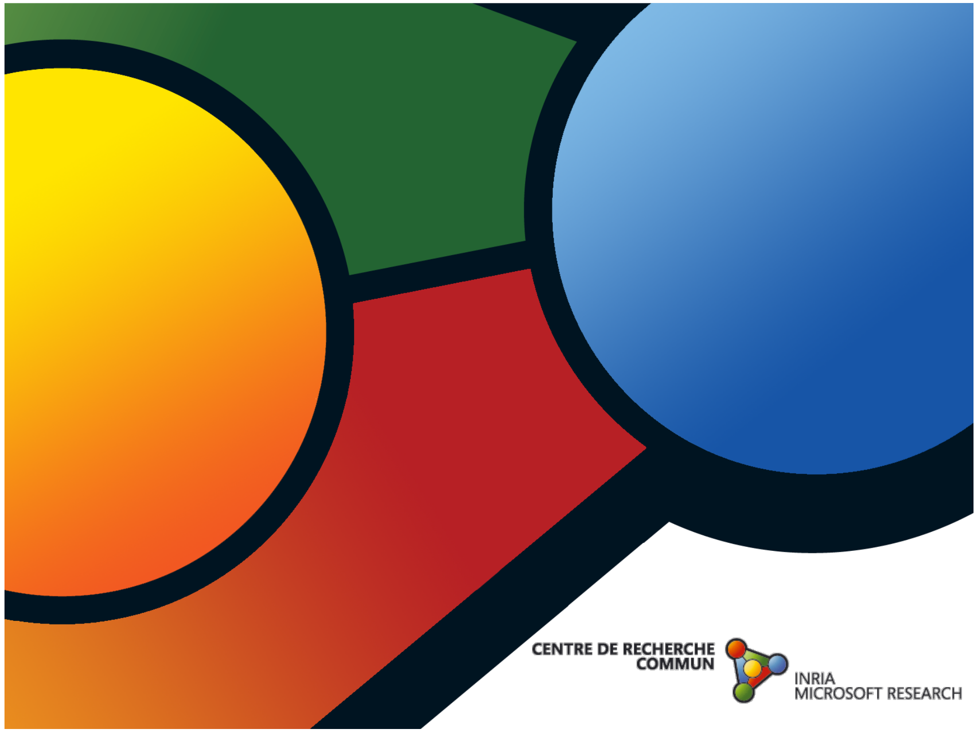
$$\begin{aligned} & (\lambda x{:}\tau\,.\,x\tau x)(\Lambda t.\lambda y{:}t.y) \\ \longrightarrow\ & (\Lambda t.\lambda y{:}t.x)\tau(\Lambda t.\lambda y{:}t.y) \\ \longrightarrow\ & (\lambda y{:}\tau.x)(\Lambda t.\lambda y{:}t.y) \\ \longrightarrow\ & (\Lambda t.\lambda y{:}t.y) \end{aligned}$$

Example

$$\Delta I \longrightarrow II \longrightarrow I \qquad A = {}^b(\forall t\,.\,{}^a({}^c t \to {}^d t))$$

$$(\lambda x{:}A\,.\,x\,A'x)(\Lambda t.\lambda y{:}t.y)$$
$$\longrightarrow (\Lambda t.\lambda y{:}t.x)A'\,(\Lambda t.\lambda y{:}t.y)$$
$$\longrightarrow (\lambda y{:}A'.x)(\Lambda t.\lambda y{:}t.y)$$
$$\longrightarrow (\Lambda t.\lambda y{:}t.y)$$

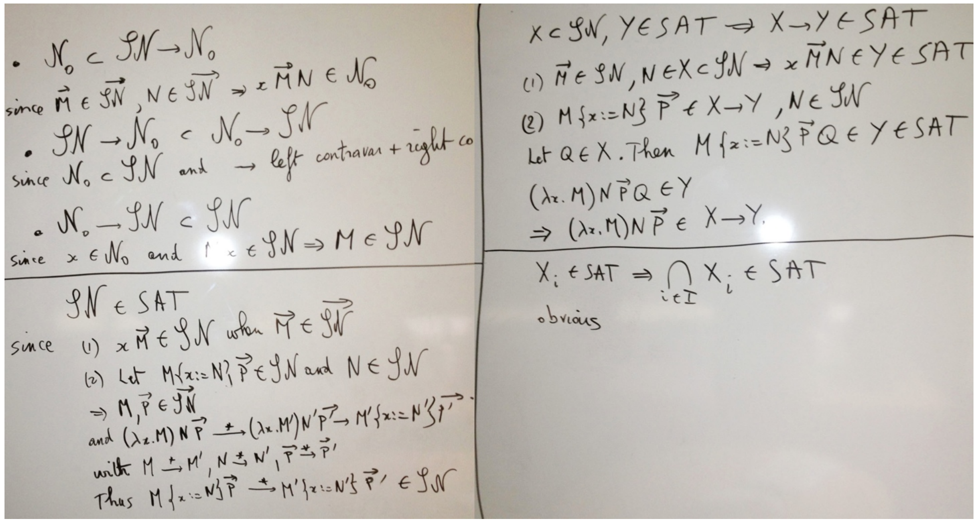
$${}^9({}^4(\lambda x{:}A\,.\,{}^3({}^1({}^0x\,A')\,{}^2x))^8(\Lambda t.{}^7(\lambda y{:}{}^5t.{}^6y)))$$
$$\longrightarrow {}^{9\lceil 4,A\rceil 3}(\,{}^1({}^{0\lfloor 4,A\rfloor 8}(\Lambda t.{}^7(\lambda y{:}{}^5t.{}^6y))A')\,{}^{2\lfloor 4,A\rfloor 8}(\Lambda t.{}^7(\lambda y{:}{}^5t.{}^6y)))$$
$$\longrightarrow {}^{9\lceil 4,A\rceil 3}(\,{}^{1\lceil 0\lfloor 4,A\rfloor 8,*\rceil 7}(\lambda y{:}\,{}^{5\lfloor 0\lfloor 4,A\rfloor 8,*\rfloor}A'.{}^6y)\,{}^{2\lfloor 4,A\rfloor 8}(\Lambda t.{}^7(\lambda y{:}{}^5t.{}^6y)))$$
$$\longrightarrow 9\lceil 4,A\rceil 3\lceil 1\lceil 0\lfloor 4,A\rfloor 8,*\rceil 7,\ {}^{5\lfloor 0\lfloor 4,A\rfloor 8,*\rfloor}A'\rceil 6\lfloor 1\lceil 0\lfloor 4,A\rfloor 8,*\rceil 7,\ {}^{5\lfloor 0\lfloor 4,A\rfloor 8,*\rfloor}A'\rfloor$$
$${}^{2\lfloor 4,A\rfloor 8}(\Lambda t.{}^7(\lambda y{:}{}^5t.{}^6y))$$



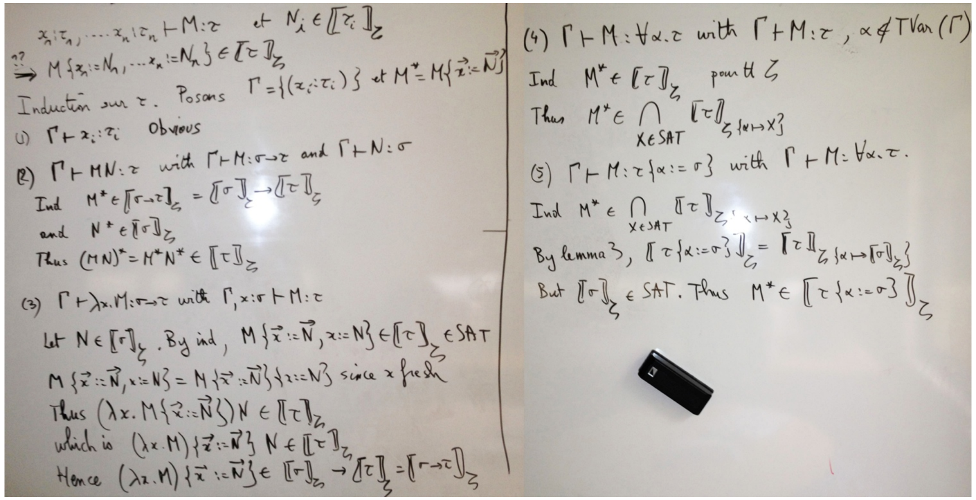
Todo list

- Relate tracking of redexes to impredicative Girard's proof
- Find intuitive argument for SN in higher-order typed λ-calculus
- Find intuitive proof for SN in higher-order typed λ-calculus
- SN proof must always be in 3rd-order Peano logic

Proofs



Proofs


x1:τ1, ... xn:τn ⊢ M:τ et Ni ∈ [[τi]]ξ
⇒? M{x1:=N1, ... xn:=Nn} ∈ [[τ]]ξ
Induction sur τ. Posons Γ = {(xi:τi)} et M* = M{x:=N}
(1) Γ ⊢ xi:τi Obvious
(2) Γ ⊢ MN:τ with Γ ⊢ M:σ→τ and Γ ⊢ N:σ
Ind M* ∈ [[σ→τ]]ξ = [[σ]]ξ → [[τ]]ξ
and N* ∈ [[σ]]ξ
Thus (MN)* = M*N* ∈ [[τ]]ξ
(3) Γ ⊢ λx.M:σ→τ with Γ, x:σ ⊢ M:τ
Let N ∈ [[σ]]ξ. By ind, M{x:=N, x:=N} ∈ [[τ]]ξ ∈ SAT
M{x:=N, x:=N} = M{x:=N}{x:=N} since x fresh
Thus (λx. M{x:=N})N ∈ [[τ]]ξ
which is (λx.M){x:=N} N ∈ [[τ]]ξ
Hence (λx.M){x:=N} ∈ [[σ]]ξ → [[τ]]ξ = [[σ→τ]]ξ
(4) Γ ⊢ M:∀α.τ with Γ ⊢ M:τ, α ∉ TVar(Γ)
Ind M* ∈ [[τ]]ξ pour tt ξ
Thus M* ∈ ⋂ X∈SAT [[τ]]ξ{α↦X}
(5) Γ ⊢ M:τ{α:=σ} with Γ ⊢ M:∀α.τ.
Ind M* ∈ ⋂ X∈SAT [[τ]]ξ{α↦X}
By lemma 3, [[τ{α:=σ}]]ξ = [[τ]]ξ{α↦[[σ]]ξ}
But [[σ]]ξ ∈ SAT. Thus M* ∈ [[τ{α:=σ}]]ξ